

# A NONLINEAR FREE BOUNDARY PROBLEM WITH A SELF-DRIVEN BERNOULLI CONDITION

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ABSTRACT. We study a Bernoulli type free boundary problem with two phases

$$J[u] = \int_{\Omega} |\nabla u(x)|^2 dx + \Phi(\mathcal{M}_-(u), \mathcal{M}_+(u)), \quad u - \bar{u} \in W_0^{1,2}(\Omega),$$

where  $\bar{u} \in W^{1,2}(\Omega)$  is a given boundary datum. Here,  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are weighted volumes of  $\{u \leq 0\} \cap \Omega$  and  $\{u > 0\} \cap \Omega$ , respectively, and  $\Phi$  is a nonnegative function of two real variables.

We show that, for this problem, the Bernoulli constant, which determines the gradient jump condition across the free boundary, is of global type and it is indeed determined by the weighted volumes of the phases.

In particular, the Bernoulli condition that we obtain can be seen as a pressure prescription in terms of the volume of the two phases of the minimizer itself (and therefore it depends on the minimizer itself and not only on the structural constants of the problem).

Another property of this type of problems is that the minimizer in  $\Omega$  is not necessarily a minimizer in a smaller subdomain, due to the nonlinear structure of the problem.

Due to these features, this problem is highly unstable as opposed to the classical case studied by Alt, Caffarelli and Friedman. It also interpolates the classical case, in the sense that the blow-up limits of  $u$  are minimizers of the Alt-Caffarelli-Friedman functional. Namely, the energy of the problem somehow linearizes in the blow-up limit.

As a special case, we can deal with the energy levels generated by the volume term  $\Phi(0, r_2) = r_2^{\frac{n-1}{n}}$ , which interpolates the Athanasopoulos-Caffarelli-Kenig-Salsa energy, thanks to the isoperimetric inequality.

In particular, we develop a detailed optimal regularity theory for the minimizers and for their free boundaries.

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## 1. INTRODUCTION

After [2, 3], a classical problem in the free boundary theory consists in studying the minimizers of an energy functional which is the *linear superposition* of a Dirichlet energy and a volume term. In this case, minimizers are proved to be harmonic away from the free boundary. Also, minimizers naturally enjoy a free boundary condition which can be seen as a balance of the normal derivatives across the interface.

This type of problems has a natural interpretation in terms of two dimensional flows of two irrotational, incompressible and inviscid fluids. Indeed, if the fluids have velocities  $\mathbf{v}^\pm = \nabla\phi^\pm$ , for some potential functions  $\phi^\pm$ , it holds that  $\Delta\phi^\pm = 0$  whenever  $\phi^\pm \neq 0$ . In addition, the Bernoulli law states that

$$(1.1) \quad \frac{p^\pm(x)}{\rho^\pm} = \frac{|\mathbf{v}^\pm|^2}{2} + C_\pm$$

along every streamline, i.e. lines for which the tangent is in the direction of the velocity (in other words the level sets of the harmonic conjugate  $\psi^\pm$  of  $\phi^\pm$ ). Here  $C_\pm$  are constants depending on the streamline and  $p^\pm$  is the pressure from either side. If the free boundary is smooth then the pressure  $p^\pm$  is continuous and therefore from Cauchy-Riemann equations, after normalization, we get that (assuming that the densities  $\rho^\pm$  are constant)

$$(1.2) \quad p^\pm(x) + \frac{|\nabla\psi^\pm|^2}{2} = C_\pm \implies |\nabla\psi^+|^2 - |\nabla\psi^-|^2 = 2(C_+ - C_-).$$

In this interpretation, we see that the free boundary condition in [2, 3] is a variational version of the classical Bernoulli law (and in fact it justifies the validity of a weak version of this law at points where the free boundary is not regular).

In this paper, we consider the case in which the energy functional is a *nonlinear superposition* of a Dirichlet energy and a volume term.

We will show that general nonlinearities may produce pathologic examples, in which minimizers may not exist, or in which the free boundary of the minimizers is not smooth. Nevertheless, under suitable structural assumptions on the nonlinearity, we will show that a sufficiently strong existence and regularity theory holds true.

In addition, we will obtain a new version of the free boundary condition, which, in our case, turns out to be of “global” type. As a matter of fact, in our case, the free boundary condition may still be seen as a balance between the normal derivatives from the two sides of the free boundary, but, differently from the classical case, this balance changes from point to point of the free boundary and the change takes into account quantities that are defined globally, and not only locally (e.g., they include the nonlinearity itself and the weighted volumes of the phases of the minimizers).

Roughly speaking, in this new free boundary condition, the quantities  $C_\pm$  in (1.1) are not constant anymore and they are not locally determined. In other words, they depend not only on the streamline but also on the weighted volumes that the streamline separates, and, above all, on the minimizers themselves: for this reason, we named this type of condition *self-driven*.

An explicit geometric example related to our problem can be given in terms of the isoperimetric inequality

$$\frac{\text{Area}(\partial\Omega^+)}{[\mathcal{L}^n(\Omega^+)]^{1-\frac{1}{n}}} \geq \frac{\text{Area}(\mathbb{S}^{n-1})}{[\mathcal{L}^n(B_1)]^{1-\frac{1}{n}}} =: c_n,$$

that is

$$\text{Area}(\partial\Omega^+) \geq c_n [\mathcal{L}^n(\Omega^+)]^{1-\frac{1}{n}}.$$

Consequently, if  $\Omega^+ := \{u > 0\}$  and  $\Phi_0(r) \leq Cr^{1-\frac{1}{n}}$  for some  $C > 0$ , we have that

$$(1.3) \quad \begin{aligned} J_{\text{ACKS}}[u] &:= \int_{\Omega} |\nabla u|^2 + \text{Area}(\partial\Omega^+) \geq \int_{\Omega} |\nabla u|^2 + c_n [\mathcal{L}^n(\Omega^+)]^{1-\frac{1}{n}} \\ &\geq \int_{\Omega} |\nabla u|^2 + c_o \Phi_0(\mathcal{L}^n(\Omega^+)) =: J[u], \end{aligned}$$

with  $c_o := c_n/C$ . In this sense, the energy functional  $J_{\text{ACKS}}$  studied in [5] provides an upper bound for the energy functional  $J$ . Notice that  $J_{\text{ACKS}}$  is a *linear interpolation* of energies (the second one being an area), while  $J$  is a *nonlinear interpolation* of energies (the second one being of volume type, but scaling like an area). The functional  $J$  in (1.3) is indeed a model case for the ones that we study in the present paper (see below for precise assumptions).

We observe that this type of problems is related to the Ginzburg-Landau model with three competing rates which balance each other for a suitable choice of the structural parameter. The exact choice of the rate gives, in the limit, the energy  $J_{\text{ACKS}}$ . Thus, in this spirit, the functional  $J$  in (1.3) describes a model in which the equilibrium is reached in terms of the best approximation of isoperimetric inequality under given constraints.

In the following subsections, we will describe the formal mathematical setting of the problem, the main results and the organization of this paper.

**1.1. Problem set-up.** Let  $\Omega \subset \mathbb{R}^n$  be an open and bounded set. In what follows,  $\lambda_1 \geq 0$  and  $\lambda_2 > 0$  are given constants. For a given measurable function  $Q : \Omega \rightarrow \mathbb{R}$ , bounded by two positive numbers

$$(1.4) \quad 0 < Q_1 \leq Q(x) \leq Q_2 < \infty,$$

we define the weighted partial volumes

$$\mathcal{M}_1(u) := \lambda_1 \int_{\Omega} Q(x) \chi_{\{u \leq 0\}}(x) dx \quad \text{and} \quad \mathcal{M}_2(u) := \lambda_2 \int_{\Omega} Q(x) \chi_{\{u > 0\}}(x) dx$$

and the total volume

$$\lambda_{\Omega} := \int_{\Omega} Q(x) dx.$$

It is easy to see that

$$(1.5) \quad \mathcal{M}_1(u) = \lambda_1 \left( \lambda_{\Omega} - \int_{\Omega} Q(x) \chi_{\{u > 0\}}(x) dx \right) = \lambda_1 (\lambda_{\Omega} - \lambda_2^{-1} \mathcal{M}_2(u)).$$

Let also  $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$ . For a given  $\Phi \in C^0(\overline{\mathbb{R}_+} \times \overline{\mathbb{R}_+}, \overline{\mathbb{R}_+}) \cap C^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$  such that  $\Phi(0, 0) = 0$ , we consider

$$(1.6) \quad \Phi_0(r) := \Phi\left(\lambda_1 (\lambda_{\Omega} - \lambda_2^{-1} r), r\right)$$

and suppose that

$$(1.7) \quad \Phi'_0(r) \geq 0 \text{ for any } r \in (0, \lambda_2 \lambda_{\Omega}).$$

In view of (1.5),

$$\Phi(\mathcal{M}_1(u), \mathcal{M}_2(u)) = \Phi_0(\mathcal{M}_2(u)).$$

In this paper we study the minimization problem of the energy functional

$$(1.8) \quad \begin{aligned} J[u] &:= \int_{\Omega} |\nabla u(x)|^2 dx + \Phi(\mathcal{M}_1(u), \mathcal{M}_2(u)) \\ &= \int_{\Omega} |\nabla u(x)|^2 dx + \Phi_0(\mathcal{M}_2(u)) \end{aligned}$$

in the admissible class

$$(1.9) \quad \mathcal{A} := \{u \in W^{1,2}(\Omega), \text{ with } u - \bar{u} \in W_0^{1,2}(\Omega)\},$$

where  $\bar{u} \in W^{1,2}(\Omega)$ .

For a given minimizer  $u$  of (1.8) the free boundary of  $u$  is denoted by  $\Gamma := \partial\Omega^+(u)$ , where

$$(1.10) \quad \Omega^+(u) := \{x \in \Omega \text{ s.t. } u(x) > 0\}.$$

This problem can be viewed as an extrapolation of the classical free boundary problem of Alt and Caffarelli [2], where the authors studied the local minimizers of the energy

$$(1.11) \quad J_{AC}[u] := \int_{\Omega} \left( |\nabla u(x)|^2 + Q(x) \chi_{\{u>0\}}(x) \right) dx.$$

Indeed, the functional in (1.8) reduces to that in (1.11) with the choices  $\lambda_1 := 0$ ,  $\lambda_2 := 1$  and  $\Phi(r_1, r_2) := r_2$ .

More generally, the functional in (1.8) is also an extrapolation of the two-phase free boundary problem in [3], in which, instead of the functional in (1.8), the minimization problem dealt with the energy

$$(1.12) \quad J_{ACF}[u] := \int_{\Omega} \left( |\nabla u(x)|^2 + \lambda_1 Q(x) \chi_{\{u \leq 0\}}(x) + \lambda_2 Q(x) \chi_{\{u > 0\}}(x) \right) dx,$$

since the functional in (1.8) reduces to that in (1.12) with the choice  $\Phi(r_1, r_2) := r_1 + r_2$  (in this case, condition (1.7) reduces to  $\lambda_2 \geq \lambda_1$ , compare with the assumptions of Theorem 2.3 in [3]).

In this sense, the energy functional in (1.8) provides a free boundary problem that is either one-phase (when  $\lambda_1 = 0$ ) or two-phase (when  $\lambda_1 \neq 0$ ) and in which the interfacial energy depends on the volume of the phases in a possibly nonlinear way, which is described by the function  $\Phi$ . The principal difference from the classical case is that the free boundary condition is determined by the weighted volumes of the phases and hence its Bernoulli constant is of global type and varies from one minimizer to another.

More precisely, if the free boundary  $\Gamma := \partial\{u > 0\}$  is a smooth hypersurface then

$$(1.13) \quad |\nabla u^+(p)|^2 - |\nabla u^-(p)|^2 = \Lambda(p), \quad p \in \Gamma,$$

where

$$(1.14) \quad \Lambda(p) := \left[ \lambda_2 \partial_{r_2} \Phi \left( \lambda_1 \int_{\Omega} Q \chi_{\{u < 0\}}, \lambda_2 \int_{\Omega} Q \chi_{\{u > 0\}} \right) - \lambda_1 \partial_{r_1} \Phi \left( \lambda_1 \int_{\Omega} Q \chi_{\{u < 0\}}, \lambda_2 \int_{\Omega} Q \chi_{\{u > 0\}} \right) \right] Q(p).$$

In the classical case  $\Lambda$  is a prescribed function and only depends on the ambient space at a given point. Conversely, since in our setting  $\Lambda$  depends in a nonlinear fashion on global quantities such as  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , which in turn depend on the solution, it is natural to expect that the problem is going to be highly unstable (other global free boundary conditions arise in unstable free boundary problems as the one dealt with in formula (1.12) in [12]). In particular, comparing (1.2) with (1.13) and (1.14), we may consider the free boundary condition of our problem as a nonlinear prescription of the pressure in terms of the volume of the two phases of the minimizer.

In our framework, the instability produced by the nonlinear superposition of energies may be, in general, quite severe, and, in fact, *the minimizers do not always exist*, as we will see in Section 3. Thus some structural assumptions are needed in order to develop a meaningful theory.

Interesting examples of nonlinearities that we can take into account are given by  $\Phi(r_1, r_2) := \frac{n-1}{r_2^n}$  and  $\Phi(r_1, r_2) := (r_1 + r_2)^{\frac{n-1}{n}}$ . We notice that this type of nonlinearities provides a scaling which is naturally induced by the isoperimetric inequality. For instance, the minimizers of the Athanasopoulos-Caffarelli-Kenig-Salsa functional [5]

$$\int_{\Omega} |\nabla u|^2 + \text{Per}_{\Omega}(\{u > 0\})$$

generate energy levels that are above the ones of our functional. If the phases have isoperimetric property then this levels coincide. In this sense, the main difference between the energy functionals in (1.8) and (1.11)–(1.12) lies in the different scaling of the volume term. This can be seen, as a paradigmatic example, by looking

at the functional

$$\int_{\Omega} |\nabla u|^2 + \Phi_0(\mathcal{L}^n(\{u > 0\} \cap \Omega))$$

with

$$cr^{\frac{n-1}{n}} \leq \Phi_0(r) \leq Cr^{\frac{n-1}{n}},$$

for some  $C \geq c > 0$ . We remark that different scalings in perimeter/volume terms may cause instability phenomena in the corresponding minimization arguments, namely a minimizer in a given domain is not necessarily a minimizer in a smaller domain, see [12].

Other cases of interest for the nonlinearity are the following ones:

- $\Phi$  only depends on the sum of the masses of the two phases, namely when  $\Phi(r_1, r_2) = \tilde{\Phi}(r_1 + r_2)$  (notice that in this case, condition (1.7) is implied by the two conditions  $\lambda_2 \geq \lambda_1$  and  $\tilde{\Phi}'(r) \geq 0$  for any  $r > 0$ ).
- $\Phi$  only depends on the the sum of different powers of the masses of the two phases, namely

$$\Phi(r_1, r_2) := \frac{r_1^{1+\alpha}}{1+\alpha} + \frac{r_2^{1-\beta}}{1-\beta},$$

with  $\alpha \geq 0$  and  $\beta > 0$ . Notice that in this case condition (1.7) is satisfied if  $\lambda_1$  is sufficiently small (possibly in dependence of  $\lambda_2$  and  $\lambda_{\Omega}$ ).

For different type of free boundary problems with a specific piecewise linear function of a volume term see [1].

**1.2. Main results.** The main results of this paper deal with the regularity of the minimizers and of their free boundary. We stress that, in general, minimizers *may not exist* and, when minimizers exist, their free boundary *may be irregular*. We will present some explicit examples of these pathologies in Sections 3 and 4.

In spite of these examples, under suitable structural assumptions, a good regularity theory holds true.

For this, recalling the notation in (1.10), we will suppose that, for a given minimizer  $u$ ,

$$(1.15) \quad \mathcal{L}^n(\Omega^+(u)) \geq \varpi > 0.$$

We will see that this assumption is not restrictive and it is satisfied in all nontrivial cases (a precise statement will be given in Lemma 5.1). Then, our first result deals with the regularity of the gradient of the minimizers in BMO spaces. To state it, for any  $x_0 \in \Omega$  and for any  $\rho > 0$ , we use the notation

$$(1.16) \quad (\nabla u)_{x_0, \rho} := \oint_{B_{\rho}(x_0)} \nabla u(x) dx.$$

Then, we have the following:

**Theorem 1.1.** *Let  $u$  be a minimizer in  $\Omega$  for the functional  $J$  in (1.8). Assume that (1.7) and (1.15) hold true.*

*Then  $\nabla u \in BMO_{\text{loc}}(\Omega, \mathbb{R}^n)$ . More precisely, for any  $D \Subset \Omega$ , there exists  $C > 0$ , possibly depending on  $\varpi$ ,  $Q$ ,  $\Omega$  and  $D$ , such that*

$$\sup_{B_r(x_0) \subseteq D} \oint_{B_r(x_0)} |\nabla u(x) - (\nabla u)_{x_0, r}| dx \leq C.$$

As a consequence of Theorem 1.1, we also obtain the following result:

**Corollary 1.2.** *Let  $u$  be a minimizer in  $\Omega$  for the functional  $J$  in (1.8). Assume that (1.7) and (1.15) hold true. Then:*

- $u$  is locally log-Lipschitz continuous, namely it is continuous, with modulus of continuity bounded by  $\sigma(t) = t |\log t|$ .
- $u$  is harmonic in the set  $\Omega^+(u)$ .

- For any  $D \Subset \Omega$ , there exists  $C > 0$ , possibly depending on  $\varpi$ ,  $Q$ ,  $\Omega$  and  $D$ , such that

$$(1.17) \quad \left| \frac{1}{r^{n-1}} \int_{\partial B_r(x_0)} u \right| \leq C r,$$

for any  $x_0 \in \Gamma$ , as long as  $B_r(x_0) \subseteq D$ .

The regularity of the minimizers can also be improved, to reach the optimal Lipschitz regularity, as given by the following result:

**Theorem 1.3.** *Let  $u$  be a minimizer in  $\Omega$  for the functional  $J$  in (1.8). Assume that (1.7) and (1.15) hold true. Then  $u \in C_{loc}^{0,1}(\Omega)$ .*

We also deal with the geometric properties of the minimizers, obtaining optimal quantitative results. In particular, we prove nondegeneracy of minimizers and linear growth from the free boundary, as stated in the next result:

**Theorem 1.4.** *Let  $u$  be a minimizer in  $\Omega$  of the energy functional  $J$  in (1.8),  $\Omega_0^+$  a connected component of the positivity set  $\Omega^+(u)$ , and  $x_0 \in \partial\Omega_0^+$ .*

*Assume that (1.7) and (1.15) hold true. Suppose that  $r > 0$  is small enough such that  $B_r(x_0) \Subset \Omega$  and*

$$(1.18) \quad \mathcal{L}^n(\Omega^+(u) \setminus B_r(x_0)) \geq \frac{\varpi}{2}.$$

*Assume also that*

$$(1.19) \quad \Theta := \inf_{[\lambda_2 Q_1 \varpi/2, \lambda_2 \lambda_\Omega]} \Phi'_0 > 0.$$

*Then, for any  $\kappa \in (0, 1)$  there exists a positive constant  $c$ , depending on  $\varpi$ ,  $\kappa$ ,  $\Theta$  and  $Q$ , such that if*

$$\int_{B_r(x_0) \cap \Omega_0^+} u^2 < c r^2,$$

*then  $u^+ = 0$  in  $B_{\kappa r}(x_0) \cap \Omega_0^+$ .*

*In particular, for any domain  $D \Subset \Omega$  there exists a positive constant  $c$ , depending only on  $\varpi$ ,  $Q$ ,  $\Omega$  and  $\text{dist}(D, \partial\Omega)$ , such that*

$$(1.20) \quad \int_{B_r(x_0) \cap \Omega_0^+} u^2 \geq c r^2,$$

*for any  $x_0 \in \partial\Omega_0^+ \cap D$  and  $r > 0$ , such that  $B_r(x_0) \Subset D$ .*

Interestingly, the result in Theorem 1.4 holds true in any connected component of the positivity set of the minimizers.

In this paper, we also establish density results for minimizers, that can be of independent interest as well, and that can be used to establish the minimizing properties of the blow-up limits of the minimizers, which indeed turn out to be minimizers of more classical free boundary problems. In this setting, the result that we obtain is the following:

**Theorem 1.5.** *Let  $u$  be a minimizer in  $\Omega$  for the functional in (1.8) and let  $u_0$  be the blow-up limit<sup>1</sup>. Assume that  $Q$  is continuous at 0 and that (1.7) and (1.15) hold true.*

*Then, for any fixed  $r > 0$ , we have that  $u_0$  is a minimizer of the functional*

$$J_0[w] := \int_{B_r(x_0)} |\nabla w|^2 + \lambda_0 \mathcal{L}^n(B_r(x_0) \cap \{w > 0\}),$$

*where*

$$(1.21) \quad \lambda_0 := \lambda_2 Q(0) \Phi'_0 \left( \lambda_2 \int_{\Omega} Q(x) \chi_{\{u > 0\}}(x) dx \right).$$

---

<sup>1</sup>A standard, explicit definition of the blow-up limit and the existence of such limit is given in Proposition 9.1.

We stress that the quantity  $\lambda_0$  in (1.21) depends on the minimizer  $u$ . Furthermore, it is useful to remark that, when  $\Phi_0$  is concave, minimizers of  $J$  are also minimizers of the functional

$$J_\star[w] := \int_{\Omega} |\nabla w|^2 + \lambda_\star \lambda_2 \int_{\Omega} Q(x) \chi_{\{w>0\}}(x) dx = \int_{\Omega} |\nabla w|^2 + \lambda_\star \mathcal{M}_2(w),$$

$$\text{with } \lambda_\star := \Phi'_0 \left( \lambda_2 \int_{\Omega} Q(x) \chi_{\{u>0\}}(x) dx \right) = \Phi'_0(\mathcal{M}_2(u)),$$

namely the nonlinear free boundary problem in (1.8) reduces to the classical ones in [2, 3] (with a coefficient depending on the minimizer itself). Indeed, if  $u$  is a minimizer of  $J$  and  $v$  is a perturbation of  $u$ , it follows that

$$\begin{aligned} 0 &\geq J[u] - J[v] \\ &= J_\star[u] - J_\star[v] + \Phi_0(\mathcal{M}_2(u)) - \Phi_0(\mathcal{M}_2(v)) + \lambda_\star [\mathcal{M}_2(v) - \mathcal{M}_2(u)]. \end{aligned}$$

Since the concavity of  $\Phi_0$  implies that

$$\Phi_0(\mathcal{M}_2(v)) - \Phi_0(\mathcal{M}_2(u)) \leq \Phi'_0(\mathcal{M}_2(u)) [\mathcal{M}_2(v) - \mathcal{M}_2(u)] = \lambda_\star [\mathcal{M}_2(v) - \mathcal{M}_2(u)],$$

we thus obtain that  $0 \geq J_\star[u] - J_\star[v]$ , hence  $u$  is a minimizer for  $J_\star$ .

In our setting, we also obtain partial regularity results (valid in any dimension) for free boundary points, as stated in the following result:

**Theorem 1.6.** *Let  $u$  be a minimizer in  $\Omega$  for the functional  $J$  in (1.8). Assume that (1.7) and (1.15) are satisfied.*

*Then, the following statements hold true:*

- (i)  $\Delta u^+$  is a Radon measure and, for any  $x \in \Gamma$  and any  $r > 0$  such that  $B_{2r}(x) \subset \Omega$  and  $\mathcal{L}^n(\Omega^+(u) \setminus B_r(x)) \geq \varpi/2$ , we have that

$$\int_{B_r(x)} \Delta u^+ \leq \frac{1}{r} \int_{B_{2r}(x)} |\nabla u^+|.$$

- (ii) For any subdomain  $D \Subset \Omega$  there exists  $r_0 > 0$  such that

$$\int_{B_r(x)} \Delta u^+ \geq cr^{n-1},$$

for any  $r \in (0, r_0)$ ,  $x \in \Gamma$  and such that  $B_r(x) \subseteq D$ , for a suitable  $c > 0$ .

- (iii) If  $B_r \subseteq \Omega$ , then

$$(1.22) \quad \mathcal{H}^{n-1}(B_r \cap \partial\{u > 0\}) < +\infty$$

$$(1.23) \quad \text{and } \mathcal{H}^{n-1}((B_r \cap \partial\{u > 0\}) \setminus \partial_{\text{red}}\{u > 0\}) = 0.$$

In dimension 2, we also obtain a complete regularity theory for the minimizers. This result goes as follows:

**Theorem 1.7.** *Let  $n = 2$ . Let  $\Phi_0$  be such that  $\Phi'_0 > 0$ . Assume also that (1.15) is satisfied. Then each free boundary point is critically flat and hence  $\partial\{u > 0\}$  is continuously differentiable.*

We stress that, in this paper, the techniques that we develop are strong enough to allow a unified treatment of the one and two phase cases simultaneously.

Furthermore, all the results of this paper are valid in any dimension (with the only exception of Lemma 11.7 and Theorem 1.7).

It is also worth to remark that many of the results presented in this paper are rather subtle to obtain, since they strongly rely on some specific behavior of the nonlinearity, and fail once these requirements are not met: for instance, the regularity result in Theorem 1.7 fails to be true in the case of nonlinear functions  $\Phi_0$  which are constant on some interval (see Theorem 1.1 in [12], where an example of free boundary given by a singular cone in the plane is constructed). In this sense, even the results whose statements “resemble” cases

already known in the literature for the linear case require a careful analysis of the different setting and a precise determination of appropriate structural assumptions.

**1.3. Organization of the paper.** The paper is organized as follows. In Section 2 we prove the existence of minimizers. The proof is standard and is based on a semicontinuity argument and on a refinement of Egoroff's theorem for Sobolev functions.

In Section 3, an explicit example of volumetric function  $\Phi_0$  for which no solution exists is constructed. While  $\Phi_0(r)$  suffers a jump at  $r = 1$ , the non-existence is still surprising because it shows that, for such  $\Phi_0$ , the set of admissible functions is not empty. In Section 4 we construct another explicit example of  $\Phi_0$  that does not satisfy the structural assumption in (1.7): in this case, minimizers do exist, but their free boundary is irregular.

That done, we begin to establish the basic properties of the minimizers in Section 5. First a lower bound for the positivity set  $\Omega^+(u)$  is proved for a suitable boundary condition. Then we show that the gradients of minimizers are locally BMO functions, that is we prove Theorem 1.1. This, in turn, implies that  $u$  is locally log-Lipschitz continuous, as given by Corollary 1.2. For the one-phase problem this immediately implies the linear growth of  $u$  away from free boundary. One of the by-products of the local BMO estimate is the coherent growth estimate (1.17). Using this and the Alt-Caffarelli-Friedman monotonicity theorem we prove the local Lipschitz regularity for the minimizers of the two-phase problem, as stated in Theorem 1.3.

Then, we use a domain variation argument, to derive the nonlocal Bernoulli condition in Section 6.

The non-degeneracy of minimizers given by Theorem 1.4 is proved in Section 7.

In Section 8 we show that for every ball  $B$  centered at the free boundary there exists a smaller ball  $B' \subset \Omega^+(u) \cap B$  such that  $\mathcal{L}^n(B') \geq c\mathcal{L}^n(B)$ , for some universal constant  $c > 0$ .

Section 9 is devoted to the study of the properties of the blow-up limits and to the proof of Theorem 1.5. In particular, we show that the blow-up limits of the minimizers become global solutions for the Alt-Caffarelli-Friedman functional provided that  $Q$  is continuous.

In Section 10 we prove the partial regularity of the free boundary, as given by Theorem 1.6, namely, that the  $\partial\{u > 0\}$  is of locally finite perimeter and the reduced boundary has full  $\mathcal{H}^{n-1}$  measure in  $\partial\{u > 0\}$ . In particular, we show that the measure theoretic normal exists at  $\mathcal{H}^{n-1}$  a.e. point of  $\partial\{u > 0\}$ .

In Section 11 we prove that at the flat free boundary points the free boundary is regular and establish the full regularity of the free boundary in two dimensions, as stated in Theorem 1.7.

## NOTATION

Let us fix some notation.

- $\mathcal{L}^n$  is the  $n$  dimensional Lebesgue measure.
- $\mathcal{H}^{n-1}$  is the  $n - 1$  dimensional Hausdorff measure.
- $u^+(x) := \max\{u(x), 0\}$  and  $u^-(x) := -\min\{u(x), 0\}$  are the positive and the negative parts of  $u$ , respectively, so that  $u = u^+ - u^-$ .
- $\lambda_1 \geq 0$  and  $\lambda_2 > 0$  are given constants.
- $\Omega^+(u) := \{x \in \Omega : u(x) > 0\}$  and  $\Omega^-(u) := \{x \in \Omega : u(x) < 0\}$  are the positivity and the negativity sets of  $u$ , respectively,
- $\varpi > 0$  is the constant providing a lower bound for the weighted volume of  $\Omega^+(u)$ .
- $\Gamma = \partial\{u > 0\}$  is the free boundary.
- The open balls are denoted by  $B_r(x_0) := \{x \in \mathbb{R}^n \text{ s.t. } |x - x_0| < r\}$  and  $B_r := B_r(0)$ .
- $C_{loc}^{0,1}(\Omega)$  is the class of locally Lipschitz continuous functions in  $\Omega$ .
- The mean value integral is  $\bar{f}_E = \frac{1}{\mathcal{L}^n(E)} \int_E f$ .
- Various universal constants are often denoted by  $C$ , for simplicity.



## 2. EXISTENCE AND BASIC PROPERTIES OF MINIMIZERS

In this section we prove that there exists  $u \in \mathcal{A}$  minimizing (1.8), where  $\mathcal{A}$  is defined in (1.9). We also show that under condition (1.7),  $u$  is globally subharmonic in  $\Omega$ .

## 2.1. Existence of minimizers.

**Lemma 2.1.** *Fix  $\bar{u} \in W^{1,2}(\Omega)$ . Then, there exists  $u \in \mathcal{A}$  such that*

$$J[u] \leq J[v]$$

for any  $v \in \mathcal{A}$ .

*Proof.* The proof is a standard lower semicontinuity argument (we give the details for the facility of the reader). We notice that

$$\mathcal{M}(u_0) \leq (\lambda_1 + \lambda_2) Q_2 \mathcal{L}^n(\Omega) < +\infty$$

and so  $J[\bar{u}] < +\infty$ . Now, let  $u_k \in \mathcal{A}$  be a minimizing sequence. We observe that  $\bar{u} \in \mathcal{A}$ , hence for sufficiently large  $k$  we may suppose that

$$(2.1) \quad J[u_k] \leq J[\bar{u}] < +\infty.$$

Set  $v_k := u_k - \bar{u} \in W_0^{1,2}(\Omega)$ . As a consequence of (2.1), we have that

$$\sup_{k \in \mathbb{N}} \|\nabla v_k\|_{L^2(\Omega, \mathbb{R}^n)} < +\infty,$$

and so, by Poincaré inequality, also

$$\sup_{k \in \mathbb{N}} \|v_k\|_{L^2(\Omega)} < +\infty.$$

Therefore, up to a subsequence,  $v_k$  converges weakly to some  $v \in W_0^{1,2}(\Omega)$ , strongly in  $L^2(\Omega)$  and a.e. in  $\Omega$ . Then,  $\nabla v_k$  converges weakly to  $\nabla v$  in  $L^2(\Omega, \mathbb{R}^n)$ . So, if we set  $u := v + \bar{u}$ , we have that  $u \in \mathcal{A}$ ,  $u_k \rightarrow u$  in  $L^2(\Omega)$  and a.e. in  $\Omega$ , and  $\nabla u_k \rightarrow \nabla u$  weakly in  $L^2(\Omega, \mathbb{R}^n)$ . In particular,

$$(2.2) \quad \liminf_{k \rightarrow +\infty} \int_{\Omega} |\nabla u_k|^2 \geq \int_{\Omega} |\nabla u|^2.$$

Now we observe that

$$(2.3) \quad \liminf_{k \rightarrow +\infty} \int_{\Omega} \lambda_2 Q(x) \chi_{\{u_k > 0\}}(x) dx \geq \int_{\Omega \setminus E_\varepsilon} \lambda_2 Q(x) \chi_{\{u > 0\}}(x) dx.$$

For this, let  $\varepsilon > 0$  be fixed. Using the refinement of Egoroff's theorem for  $W^{1,2}$  functions, it follows that there exists a subset  $E_\varepsilon \subset \Omega$  such that  $u_k \rightarrow u$  uniformly in  $\Omega \setminus E_\varepsilon$  and  $\text{cap}_2(E_\varepsilon) < \varepsilon$  where  $\text{cap}_2$  is the 2-capacity. Thus

$$\liminf_{k \rightarrow +\infty} \int_{\Omega} \lambda_2 Q(x) \chi_{\{u_k > 0\}}(x) dx \geq \int_{\Omega \setminus E_\varepsilon} \lambda_2 Q(x) \chi_{\{u > 0\}}(x) dx - \lambda_2 Q_2 |E_\varepsilon|.$$

Sending  $\varepsilon \rightarrow 0$  the result in (2.3) follows.

From (2.3), we deduce that

$$(2.4) \quad \liminf_{k \rightarrow +\infty} \mathcal{M}_2(u_k) = \liminf_{k \rightarrow +\infty} \int_{\Omega} \lambda_2 Q(x) \chi_{\{u_k > 0\}}(x) dx \geq \int_{\Omega} \lambda_2 Q(x) \chi_{\{u > 0\}}(x) dx = \mathcal{M}_2(u).$$

Furthermore, by (1.5),

$$0 \leq \mathcal{M}_1(u) = \lambda_1 (\lambda_\Omega - \lambda_2^{-1} \mathcal{M}_2(u))$$

and therefore  $\mathcal{M}_2(u) \in [0, \lambda_2 \lambda_\Omega]$  and, similarly,  $\mathcal{M}_2(u_k) \in [0, \lambda_2 \lambda_\Omega]$ .

Therefore, since, by (1.7), the function  $\Phi_0$  is nondecreasing in  $[0, \lambda_2 \lambda_\Omega]$ , we deduce from (2.4) that

$$\liminf_{k \rightarrow +\infty} \Phi_0(\mathcal{M}_2(u_k)) = \Phi_0 \left( \liminf_{k \rightarrow +\infty} \mathcal{M}_2(u_k) \right) \geq \Phi_0(\mathcal{M}_2(u)).$$

This and (2.2) give that

$$\liminf_{k \rightarrow +\infty} J[u_k] \geq J[u],$$

hence  $u$  is the desired minimizer.  $\square$

**2.2. Euler-Lagrange equations.** We now state the basic properties of the minimizers. The starting point is to derive the differential inequalities that the minimizers satisfy in  $\Omega$ .

**Lemma 2.2.** *Let  $u$  be a minimizer in  $\Omega$  for the functional  $J$  in (1.8) and suppose that (1.7) holds true. Then  $u$  is subharmonic.*

*Proof.* We use some classical ideas in Lemma 2.2 of [2] and Theorem 2.3 in [3], combining them here with condition (1.7). For this, we consider a ball  $B \Subset \Omega$  and the function  $v$  which is harmonic in  $B$  and coincides with  $u$  in  $\Omega \setminus B$ . We also take  $w := \min\{u, v\}$ . Then,  $w$  is an admissible competitor for  $u$  and therefore  $J[u] \leq J[w]$ , that is

$$(2.5) \quad I := \int_B |\nabla u(x)|^2 dx - \int_B |\nabla w(x)|^2 dx \leq \Phi_0(\mathcal{M}_2(w)) - \Phi_0(\mathcal{M}_2(u)).$$

On the other hand, if we set  $z := \max\{u - v, 0\}$ , we have that

$$\begin{aligned} I &= \int_B (\nabla(u - w)(x)) \cdot (\nabla(u + w)(x)) dx = \int_{B \cap \{u > v\}} (\nabla(u - v)(x)) \cdot (\nabla(u + v)(x)) dx \\ &= \int_{B \cap \{u > v\}} |\nabla(u - v)(x)|^2 dx + 2 \int_{B \cap \{u > v\}} (\nabla(u - v)(x)) \cdot \nabla v(x) dx \\ &= \int_{B \cap \{u > v\}} |\nabla(u - v)(x)|^2 dx = \int_B |\nabla z(x)|^2 dx. \end{aligned}$$

Inserting this into (2.5), we obtain that

$$(2.6) \quad \int_B |\nabla z(x)|^2 dx \leq \Phi_0(\mathcal{M}_2(w)) - \Phi_0(\mathcal{M}_2(u)).$$

Moreover, we have that  $w \leq u$  and therefore  $\chi_{\{w > 0\}} \leq \chi_{\{u > 0\}}$ . Accordingly,  $\mathcal{M}_2(w) \leq \mathcal{M}_2(u)$  and then, in light of (1.7), we obtain that  $\Phi_0(\mathcal{M}_2(w)) \leq \Phi_0(\mathcal{M}_2(u))$ . From this and (2.6) we deduce that  $z$  is constant in  $B$ . Since  $z$  vanishes in  $\Omega \setminus B$ , we conclude that  $z$  vanishes in  $\Omega$  and therefore that  $u \leq v$ , which establishes the desired result.  $\square$

### 3. NON EXISTENCE OF MINIMIZERS FOR IRREGULAR NONLINEARITIES

In this section, we observe that when the regularity and the structural assumptions on  $\Phi_0$  are violated, minimizers may not exist. To exhibit this phenomenon in an explicit and concrete example, we consider the case in which  $\Omega := (0, 1) \subset \mathbb{R}$ ,  $\lambda_2 := Q := 1$ ,  $\bar{u}(x) := x$  for any  $x \in [0, 1]$ , and

$$\Phi(r_1, r_2) := \begin{cases} r_2 & \text{if } r_2 \in [0, \frac{1}{2}], \\ \frac{5-2r_2}{8} & \text{if } r_2 \in (\frac{1}{2}, 1), \\ 1 & \text{if } r_2 \in [1, +\infty). \end{cases}$$

Notice that, with this setting,

$$(3.1) \quad \Phi_0(r) = \begin{cases} r & \text{if } r \in [0, \frac{1}{2}], \\ \frac{5-2r}{8} & \text{if } r \in (\frac{1}{2}, 1), \\ 1 & \text{if } r \in [1, +\infty). \end{cases}$$

For this choice of  $\Phi_0$ , *there exists no minimizer  $u^*$  for the energy functional in (1.8) with the condition that  $u^* - \bar{u} \in W_0^{1,2}(\Omega)$ .*

To see this, let us suppose, by contradiction, that such minimizer exists. Then,

$$(3.2) \quad \int_{\Omega} |\dot{u}^*|^2 \leq J[u^*] \leq J[\bar{u}] = 1 + \Phi_0(1) = 2.$$

As a consequence,

$$1 - 0 = u^*(1) - u^*(0) = \int_{(0,1) \cap \{u^* > 0\}} \dot{u}^* \leq \sqrt{\int_{\Omega} |\dot{u}^*|^2} \sqrt{\mathcal{L}^1((0,1) \cap \{u^* > 0\})} \leq \sqrt{2} \sqrt{\mathcal{M}_2(u^*)}$$

and so

$$(3.3) \quad \mathcal{M}_2(u^*) \geq \frac{1}{2}.$$

We claim that

$$(3.4) \quad \{u^* = 0\} \text{ has positive measure.}$$

To check this, we argue by contradiction and assume that  $\mathcal{L}^1((0,1) \cap \{u^* = 0\}) = 0$ , hence  $\mathcal{M}_2(u^*) = 1$ . Consequently, since  $\bar{u}$  is a minimizer for the Dirichlet energy in  $(0,1)$ , we find that

$$(3.5) \quad J[u^*] = \int_0^1 |\dot{u}^*|^2 + 1 \geq \int_0^1 |\dot{\bar{u}}|^2 + 1 = 2.$$

Now we define, for any  $\delta \in (0, \frac{1}{2})$ ,

$$u_{\delta}(x) := \begin{cases} 0 & \text{if } x \in [0, \delta], \\ \frac{x-\delta}{1-\delta} & \text{if } x \in (\delta, 1]. \end{cases}$$

Then, there holds

$$J[u^*] \leq J[u_{\delta}] = \int_{\delta}^1 \left| \frac{1}{1-\delta} \right|^2 + \Phi_0(1-\delta) = \frac{1}{1-\delta} + \frac{5-2(1-\delta)}{8}.$$

Accordingly, by taking  $\delta$  as small as we wish, we obtain

$$J[u^*] \leq 1 + \frac{3}{8}.$$

This inequality is in contradiction with (3.5) and so it proves (3.4).

In particular, from (3.4), we can take a Lebesgue point  $p \in (0,1)$  for  $\{u^* = 0\}$ . Thus, if  $\varepsilon > 0$  is sufficiently small, we have that

$$(3.6) \quad \mathcal{L}^1((p-\varepsilon, p+\varepsilon) \cap \{u^* = 0\}) \geq \varepsilon.$$

For small  $\varepsilon > 0$ , we can also suppose that  $(p-\varepsilon, p+\varepsilon) \subset (0,1)$ .

Now we take  $\varphi \in C_0^{\infty}([-1,1])$  with  $\varphi > 0$  in  $(-1,1)$  and  $|\varphi| \leq 1$ . For any  $\varepsilon > 0$ , we define  $\varphi_{\varepsilon}(x) := \varphi(\frac{x-p}{\varepsilon})$  and we remark that

$$(3.7) \quad \int_{p-\varepsilon}^{p+\varepsilon} |\dot{\varphi}_{\varepsilon}|^2 \leq \frac{2}{\varepsilon}.$$

Let also

$$u_{\varepsilon}(x) := u^*(x) + \varepsilon^4 \varphi_{\varepsilon}(x).$$

Notice that  $u_{\varepsilon} \geq u^*$ , and

$$(3.8) \quad \begin{aligned} \mathcal{M}_2(u_{\varepsilon}) - \mathcal{M}_2(u^*) &= \mathcal{L}^1((0,1) \cap \{u_{\varepsilon} > 0\}) - \mathcal{L}^1((0,1) \cap \{u^* > 0\}) \\ &= \mathcal{L}^1(\{x \in (p-\varepsilon, p+\varepsilon) \text{ s.t. } u^*(x) = 0\}) \in [\varepsilon, 2\varepsilon], \end{aligned}$$

thanks to (3.6). Notice also that, in view of (3.3) and (3.4),

$$\mathcal{M}_2(u^*) \in \left[ \frac{1}{2}, 1 \right)$$

and so, if  $\varepsilon > 0$  is sufficiently small, we deduce from (3.8) that also

$$\mathcal{M}_2(u_\varepsilon) \in \left[ \frac{1}{2}, 1 \right).$$

Therefore, by (3.1) and (3.8),

$$(3.9) \quad \Phi_0(\mathcal{M}_2(u_\varepsilon)) - \Phi_0(\mathcal{M}_2(u^*)) = -\frac{\mathcal{M}_2(u_\varepsilon) - \mathcal{M}_2(u^*)}{4} \leq -\frac{\varepsilon}{4}.$$

On the other hand, recalling (3.2) and (3.7),

$$\begin{aligned} \int_{\Omega} |\dot{u}_\varepsilon|^2 - \int_{\Omega} |\dot{u}^*|^2 &= \int_{p-\varepsilon}^{p+\varepsilon} (2\varepsilon^4 \dot{u}^* \dot{\varphi}_\varepsilon + \varepsilon^8 |\dot{\varphi}_\varepsilon|^2) \\ &\leq 2\varepsilon^4 \sqrt{\int_{\Omega} |\dot{u}^*|^2} \sqrt{\int_{p-\varepsilon}^{p+\varepsilon} |\dot{\varphi}_\varepsilon|^2} + \varepsilon^8 \int_{p-\varepsilon}^{p+\varepsilon} |\dot{\varphi}_\varepsilon|^2 \leq 2\varepsilon^4 \sqrt{2} \sqrt{\frac{2}{\varepsilon}} + 2\varepsilon^7 \leq \varepsilon^3, \end{aligned}$$

as soon as  $\varepsilon > 0$  is small enough. Using this and (3.9), we obtain that

$$J[u_\varepsilon] - J[u^*] \leq -\frac{\varepsilon}{4} + \varepsilon^3 < 0$$

if  $\varepsilon > 0$  is small enough, which is a contradiction with the minimality of  $u^*$ . This shows that no minimizer exists in this case.

#### 4. IRREGULAR FREE BOUNDARIES

In this section, we would like to remark that if  $\Phi_0$  is not monotone, then there may exist minimizers whose free boundary is not regular, even in dimension 2 (therefore, the result in Theorem 1.7 cannot be generalized to nonlinear problems for which  $\Phi_0$  is not monotone).

To make an explicit example, we consider the case in which  $n = 2$ ,  $\Omega := B_1 \subset \mathbb{R}^2$ ,  $\lambda_1 := \lambda_2 := Q := 1$  and  $\bar{u}(x) := x_1 x_2$ . We also define

$$\begin{aligned} c_1 &:= \int_{\partial B_1} \bar{u}^+, \\ c_2 &:= \frac{c_1}{2} \left[ \int_{B_1} |\nabla \bar{u}|^2 + 1 \right]^{-1}, \\ c_3 &:= 2 + \frac{1}{4c_2} \\ \text{and} \quad c_\star &:= \min \left\{ \frac{\pi}{4}, \frac{c_1}{2c_3} \right\}. \end{aligned}$$

We remark that  $c_\star < \pi/2$ . We consider a smooth function  $\phi_\star : [0, +\infty) \rightarrow [0, +\infty)$  with  $\phi_\star(0) = 0$ ,  $\phi_\star(r) > 0$  for any  $r > 0$ ,  $\phi_\star(\pi/4) = 2$  and

$$(4.1) \quad 1 = \phi_\star(\pi/2) = \min_{[c_\star, +\infty)} \phi_\star.$$

Let also  $\Phi(r_1, r_2) := \phi_\star(r_2)$ . In this way, we have that  $\Phi_0(r) = \phi_\star(r)$  and we observe that all our structural assumptions on  $\Phi_0$  are satisfied in this case, except the monotonicity.

We will show that

$$(4.2) \quad J[\bar{u}] = \min_{u - \bar{u} \in W_0^{1,2}(B_1)} J[u],$$

hence  $\bar{u}$  is a minimizer for  $J$  in  $B_1$  with respect to its own boundary values. Interestingly, the set  $\{\bar{u} > 0\}$  is in this case a singular cone, which shows that the monotonicity assumption on  $\Phi_0$  cannot be dropped if one wishes to prove that the free boundary of minimizers in the plane is smooth.

To prove (4.2), we argue as follows. Let  $u$  be such that  $u - \bar{u} \in W_0^{1,2}(B_1)$  and set  $v := \min\{u, 1\}$ . Then,  $v - \bar{u} \in W_0^{1,2}(B_1)$ , therefore  $v = \bar{u}$  along  $\partial B_1$  and thus, if  $\nu$  is the exterior normal of  $B_1$ ,

$$(4.3) \quad \begin{aligned} c_1 &= \int_{\partial B_1} \bar{u}^+ = \int_{\partial B_1} v^+ = \int_{\partial B_1} v^+ x \cdot \nu = \int_{B_1} \operatorname{div}(v^+ x) = \int_{B_1} (2v^+ + \nabla v^+ \cdot x) \\ &\leq 2\mathcal{L}^2(B_1 \cap \{v > 0\}) + \int_{B_1} |\nabla v^+|. \end{aligned}$$

Now we observe that

$$\begin{aligned} \int_{B_1} |\nabla v^+| &= \int_{B_1} 2 \cdot \frac{1}{2\sqrt{c_2}} \cdot \sqrt{c_2} |\nabla v^+| \leq \int_{B_1 \cap \{v > 0\}} \left( \frac{1}{4c_2} + c_2 |\nabla v|^2 \right) \\ &\leq \frac{1}{4c_2} \mathcal{L}^2(B_1 \cap \{v > 0\}) + c_2 \int_{B_1} |\nabla u|^2 \leq (c_3 - 2) \mathcal{L}^2(B_1 \cap \{v > 0\}) + c_2 J[u]. \end{aligned}$$

By plugging this information into (4.3) and recalling that  $\{v > 0\} = \{u > 0\}$ , we obtain

$$c_1 \leq c_3 \mathcal{L}^2(B_1 \cap \{u > 0\}) + c_2 J[u].$$

This implies that either

$$(4.4) \quad c_3 \mathcal{L}^2(B_1 \cap \{u > 0\}) \geq \frac{c_1}{2}$$

or

$$(4.5) \quad c_2 J[u] \geq \frac{c_1}{2}.$$

If (4.4) is satisfied, then

$$\mathcal{M}_2(u) = \mathcal{L}^2(B_1 \cap \{u > 0\}) \geq \frac{c_1}{2c_3} \geq c_\star.$$

Consequently, by (4.1) and using that  $\pi = \mathcal{L}^2(B_1)$ , we have

$$\Phi_0(\mathcal{M}_2(u)) \geq 1 = \Phi_0(\pi/2) = \Phi_0(\mathcal{M}_2(\bar{u})).$$

Therefore, since  $\bar{u}$  is harmonic, we conclude that  $J[u] \geq J[\bar{u}]$ . This proves (4.2) in this case, and we now consider the case in which (4.5) holds true.

In this setting, we have that

$$J[u] \geq \frac{c_1}{2c_2} = \int_{B_1} |\nabla \bar{u}|^2 + 1 = \int_{B_1} |\nabla \bar{u}|^2 + \Phi_0(\pi/2) = J[\bar{u}].$$

This proves (4.2), as<sup>2</sup> desired.

## 5. BMO GRADIENT ESTIMATES AND LIPSCHITZ CONTINUITY OF THE MINIMIZERS

In this section, we will prove that minimizers have gradient which is locally in BMO and, as a consequence, we obtain an estimate for the integral averages of the minimizers. This method is structurally quite different from the classical techniques in [2, 3], which obtain Lipschitz estimates in the linear case without using BMO estimates on the gradient of the solution.

From now on we assume that (1.7) and (1.15) hold true.

We remark that condition (1.15) is satisfied in all nontrivial cases, and then it links  $\varpi$  to quantities only depending on  $\Omega$  and  $u_0$ . More precisely, condition (1.15) is satisfied provided that  $\bar{u}^+ = u^+|_{\partial\Omega}$  has some positive mass along the boundary (and when this does not happen, the positive phase of the minimizer is trivial). Indeed, we have the following observation:

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<sup>2</sup>Let us remark that the counterexamples discussed here are based on the intuition that when  $\Phi'_0$  vanishes at a minimizer, then the problem is related to that of harmonic functions, and so the regularity of the free boundary may be violated by looking at harmonic functions with irregular level sets. Notice also that the condition  $\Phi'_0 < 0$  reduces to  $\Phi'_0 > 0$  by replacing  $u$  by  $-u$ . We think that it is a very interesting problem to further investigate the cases in which  $\Phi_0$  is not differentiable, or not continuous.

**Lemma 5.1.** *Let  $u$  be a minimizer in  $\Omega$  for the functional  $J$  in (1.8). Assume that  $\Omega$  has Lipschitz boundary and that  $\bar{u}^+$  has some positive mass along  $\partial\Omega$ . Then (1.15) is satisfied, with*

$$\varpi := \frac{1}{C_\Omega \left( C_\Omega J[\bar{u}] + 2\|\bar{u}^+\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} \bar{u}^+ \right)} \left( \int_{\partial\Omega} \bar{u}^+ \right)^2,$$

where  $C_\Omega > 0$  is the trace constant for the domain  $\Omega$  for the embedding of  $W^{1,1}(\Omega)$  into  $L^1(\partial\Omega)$ .

*Proof.* First of all, Lemma 2.2 gives that  $u$  is subharmonic, hence so is  $u^+$ . Therefore

$$\|u^+\|_{L^\infty(\Omega)} = \|u^+\|_{L^\infty(\partial\Omega)} = \|\bar{u}^+\|_{L^\infty(\partial\Omega)}.$$

Moreover, by the minimality of  $u$ ,

$$\int_{\Omega^+(u)} |\nabla u|^2 \leq \int_{\Omega} |\nabla u|^2 \leq J[u] \leq J[\bar{u}].$$

Now, let  $\eta > 0$ , to be chosen appropriately. By the trace inequality (see e.g. Theorem 1(ii) on page 258 of [13]) and the observations above, we have that

$$\begin{aligned} \int_{\partial\Omega} \bar{u}^+ &= \|u^+\|_{L^1(\partial\Omega)} \leq C_\Omega \|u^+\|_{W^{1,1}(\Omega)} \\ &= C_\Omega \int_{\Omega^+(u)} (|\nabla u| + u) \leq C_\Omega \int_{\Omega^+(u)} \left( \eta |\nabla u|^2 + \frac{1}{4\eta} + \|u^+\|_{L^\infty(\Omega)} \right) \\ &\leq C_\Omega \left[ \eta J[\bar{u}] + \left( \frac{1}{4\eta} + \|\bar{u}^+\|_{L^\infty(\partial\Omega)} \right) \mathcal{L}^n(\Omega^+(u)) \right]. \end{aligned}$$

Hence, we choose

$$\eta := \frac{\int_{\partial\Omega} \bar{u}^+}{2C_\Omega J[\bar{u}]}$$

and we obtain that

$$C_\Omega \left( \frac{C_\Omega J[\bar{u}]}{2 \int_{\partial\Omega} \bar{u}^+} + \|\bar{u}^+\|_{L^\infty(\partial\Omega)} \right) \mathcal{L}^n(\Omega^+(u)) \geq \frac{1}{2} \int_{\partial\Omega} \bar{u}^+,$$

which gives the desired result.  $\square$

Now we establish the BMO estimate claimed in Theorem 1.1:

*Proof of Theorem 1.1.* Let  $B_r(x_0) \subseteq D \Subset \Omega$ . Without loss of generality, we assume that  $u$  does not vanish identically, hence  $\mathcal{L}^n(\Omega^+(u)) > 0$ . We consider here the function  $v \in W^{1,2}(B_r(x_0))$  which solves

$$(5.1) \quad \begin{cases} \Delta v = 0 & \text{in } B_r(x_0), \\ v = u & \text{on } \partial B_r(x_0). \end{cases}$$

We also extend  $v$  to be equal to  $u$  in  $\Omega \setminus B_r(x_0)$ . Since  $v \in \mathcal{A}$ , we have that  $J[u] \leq J[v]$  and therefore

$$\begin{aligned} (5.2) \quad \int_{B_r(x_0)} (|\nabla u|^2 - |\nabla v|^2) &\leq \Phi_0(\mathcal{M}_2(v)) - \Phi_0(\mathcal{M}_2(u)) \\ &= \Phi_0 \left( \lambda_2 \int_{\Omega \setminus B_r(x_0)} Q\chi_{\{u>0\}} + \lambda_2 \int_{B_r(x_0)} Q\chi_{\{v>0\}} \right) \\ &\quad - \Phi_0 \left( \lambda_2 \int_{\Omega \setminus B_r(x_0)} Q\chi_{\{u>0\}} + \lambda_2 \int_{B_r(x_0)} Q\chi_{\{u>0\}} \right). \end{aligned}$$

Now we claim that

$$(5.3) \quad \int_{B_r(x_0)} (|\nabla u|^2 - |\nabla v|^2) \leq C_\star r^n,$$

for some  $C_\star > 0$  (independent of  $r$ ). To prove this, we first assume that  $r > 0$  is so small that

$$(5.4) \quad \int_{\Omega \setminus B_r(x_0)} \chi_{\{u>0\}} \geq \frac{\mathcal{L}^n(\Omega^+(u))}{2} =: c_\star.$$

We stress that  $c_\star > 0$  depends on  $\varpi$  (but not on  $r$ ). We also set

$$C_0 := \sup_{\xi \in [\lambda_2 c_\star, \lambda_2 \lambda_\Omega]} \Phi'_0(\xi).$$

Let us fix  $a \geq \lambda_2 c_\star$  and  $b, c \in [0, +\infty)$ , with  $b \geq c$  and  $a + b < \lambda_2 \lambda_\Omega$ , then observe that

$$\Phi_0(a+b) - \Phi_0(a+c) = \int_c^b \Phi'_0(a+\tau) d\tau \leq C_0 (b-c) \leq C_0 b.$$

On the other hand, if  $a \geq \lambda_2 c_\star$  and  $b, c \in [0, +\infty)$ , with  $b \leq c$  and  $a + c < \lambda_2 \lambda_\Omega$ , then we have that

$$\Phi_0(a+b) - \Phi_0(a+c) \leq 0,$$

due to (1.7). Therefore, for any  $a \geq \lambda_2 c_\star$  and  $b, c \in [0, +\infty)$ , with  $a+b, a+c < \lambda_2 \lambda_\Omega$ , we get

$$\Phi_0(a+b) - \Phi_0(a+c) \leq C_0 b.$$

Utilizing this inequality with

$$(5.5) \quad \begin{aligned} a &:= \lambda_2 \int_{\Omega \setminus B_r(x_0)} Q \chi_{\{u>0\}}, \\ b &:= \lambda_2 \int_{B_r(x_0)} Q \chi_{\{v>0\}} \\ \text{and } c &:= \lambda_2 \int_{B_r(x_0)} Q \chi_{\{u>0\}} \end{aligned}$$

yields

$$\begin{aligned} &\Phi_0 \left( \lambda_2 \int_{\Omega \setminus B_r(x_0)} Q \chi_{\{u>0\}} + \lambda_2 \int_{B_r(x_0)} Q \chi_{\{v>0\}} \right) - \Phi_0 \left( \lambda_2 \int_{\Omega \setminus B_r(x_0)} Q \chi_{\{u>0\}} + \lambda_2 \int_{B_r(x_0)} Q \chi_{\{u>0\}} \right) \\ &\leq C_0 \lambda_2 \int_{B_r(x_0)} Q \chi_{\{v>0\}} \leq C_\star r^n, \end{aligned}$$

for some  $C_\star > 0$  (possibly depending on  $\varpi$  in (1.15)). Plugging this into (5.2) we see that (5.3) is satisfied if  $r$  is chosen so small to fulfill (5.4) (say  $r \in [0, r_0]$ ).

Now we complete the proof of (5.3) when  $r > r_0$ . In this case, we use the notation in (5.5) and we claim that

$$(5.6) \quad \Phi_0(a+b) \leq \hat{C} (b+1),$$

for some  $\hat{C} > 0$ , possibly depending on  $r_0, Q, \lambda_2$  and  $\Omega$ . To this goal, we define

$$a_0 := \lambda_2 \int_{\Omega \setminus B_{r_0}(x_0)} Q \quad \text{and} \quad b_0 := a + b - a_0.$$

Notice that the condition  $r > r_0$  implies that  $a_0 \geq a$  and so  $b_0 \leq b$ . We distinguish two cases, either  $b_0 \leq 0$  or  $b_0 > 0$ . If  $b_0 \leq 0$ , we use (1.7) and we obtain

$$\Phi_0(a+b) = \Phi_0(a_0+b_0) \leq \Phi_0(a_0).$$

This implies (5.6) in this case. If instead  $b_0 > 0$ , we have

$$\Phi_0(a+b) = \Phi_0(a_0+b_0) = \int_0^{b_0} \Phi'_0(a_0+\tau) d\tau + \Phi_0(a_0) \leq \sup_{\xi \in [a_0, \lambda_2 \lambda_\Omega]} \Phi'_0(\xi) b_0 + \Phi_0(a_0) \leq \sup_{\xi \in [a_0, \lambda_2 \lambda_\Omega]} \Phi'_0(\xi) b + \Phi_0(a_0),$$

which completes the proof of (5.6).

Now we observe that  $b \leq C' r^n$ , for some  $C' > 0$ . This fact, together with (5.6) and the assumption that  $r > r_0$ , gives that  $\Phi_0(a+b) \leq \tilde{C} r^n$ . Using this and the estimate in (5.2), we find that

$$\int_{B_r(x_0)} (|\nabla u|^2 - |\nabla v|^2) \leq \Phi_0 \left( \lambda_2 \int_{\Omega \setminus B_r(x_0)} Q \chi_{\{u>0\}} + \lambda_2 \int_{B_r(x_0)} Q \chi_{\{v>0\}} \right) = \Phi_0(a+b) \leq \tilde{C} r^n,$$

which establishes (5.3) also when  $r > r_0$ .

In addition, by (5.1),

$$\int_{B_r(x_0)} (|\nabla u|^2 - |\nabla v|^2) = \int_{B_r(x_0)} (|\nabla u|^2 - |\nabla v|^2 + 2\nabla v \cdot \nabla(v-u)) = \int_{B_r(x_0)} |\nabla u - \nabla v|^2.$$

This and (5.3) yield that

$$(5.7) \quad \int_{B_r(x_0)} |\nabla u - \nabla v|^2 \leq C_* r^n.$$

Now we use some techniques developed in [11]. We recall the notation in (1.16) and we observe that, using Hölder inequality,

$$(5.8) \quad \begin{aligned} \int_{B_r(x_0)} |(\nabla v)_{x_0,r} - (\nabla u)_{x_0,r}|^2 &= \int_{B_r(x_0)} \left| \frac{1}{\mathcal{L}^n(B_r(x_0))} \left( \int_{B_r(x_0)} \nabla v - \nabla u \right) \right|^2 \\ &\leq \frac{1}{\mathcal{L}^n(B_r(x_0))} \left( \int_{B_r(x_0)} |\nabla v - \nabla u| \right)^2 \leq \int_{B_r(x_0)} |\nabla v - \nabla u|^2. \end{aligned}$$

Furthermore, we recall the following Campanato growth type estimate (see e.g. Theorem 5.1 in [10]), valid for any  $0 < r < R$ ,

$$(5.9) \quad \int_{B_r(x_0)} |\nabla v - (\nabla v)_{x_0,r}|^2 \leq C \left( \frac{r}{R} \right)^{n+\alpha} \int_{B_R(x_0)} |\nabla v - (\nabla v)_{x_0,R}|^2,$$

for suitable  $\alpha \in (0, 1)$  and  $C > 1$ .

Now, using (5.8) and possibly allowing  $C$  to be a universal constant varying from line to line, we have

$$\begin{aligned} &\int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^2 \\ &\leq C \left[ \int_{B_r(x_0)} |\nabla u - \nabla v|^2 + \int_{B_r(x_0)} |\nabla v - (\nabla v)_{x_0,r}|^2 + \int_{B_r(x_0)} |(\nabla v)_{x_0,r} - (\nabla u)_{x_0,r}|^2 \right] \\ &\leq C \left[ \int_{B_r(x_0)} |\nabla u - \nabla v|^2 + \int_{B_r(x_0)} |\nabla v - (\nabla v)_{x_0,r}|^2 \right]. \end{aligned}$$

So, using (5.9),

$$(5.10) \quad \int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^2 \leq C \left[ \int_{B_r(x_0)} |\nabla u - \nabla v|^2 + \left( \frac{r}{R} \right)^{n+\alpha} \int_{B_R(x_0)} |\nabla v - (\nabla v)_{x_0,R}|^2 \right].$$

Now we remark that

$$\begin{aligned} &\int_{B_R(x_0)} |\nabla v - (\nabla v)_{x_0,R}|^2 \\ &\leq C \left[ \int_{B_R(x_0)} |\nabla v - \nabla u|^2 + \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2 + \int_{B_R(x_0)} |(\nabla u)_{x_0,R} - (\nabla v)_{x_0,R}|^2 \right] \\ &\leq C \left[ \int_{B_R(x_0)} |\nabla v - \nabla u|^2 + \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2 \right], \end{aligned}$$

where (5.8) has been used once again.



Let us plug this into (5.10), and recall that  $r \leq R$ . We exploit (5.7), and conclude that

$$\begin{aligned}
& \int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^2 \\
& \leq C \left[ \int_{B_r(x_0)} |\nabla u - \nabla v|^2 + \left(\frac{r}{R}\right)^{n+\alpha} \int_{B_R(x_0)} |\nabla v - \nabla u|^2 + \left(\frac{r}{R}\right)^{n+\alpha} \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2 \right] \\
& \leq C \left[ \int_{B_R(x_0)} |\nabla u - \nabla v|^2 + \left(\frac{r}{R}\right)^{n+\alpha} \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2 \right] \\
& \leq CR^n + C \left(\frac{r}{R}\right)^{n+\alpha} \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2.
\end{aligned}$$

Therefore, defining

$$\psi(r) := \sup_{t \leq r} \int_{B_t(x_0)} |\nabla u - (\nabla u)_{x_0,t}|^2,$$

we have that

$$\psi(r) \leq CR^n + C \left(\frac{r}{R}\right)^{n+\alpha} \psi(R).$$

Thus, by Lemma 2.1 in Chapter 3 of [14], we conclude that there exist  $c > 0$  and  $R_0 > 0$  such that

$$\psi(r) \leq Cr^n \left( \frac{\psi(R)}{R^n} + 1 \right)$$

for all  $r \leq R \leq R_0$ , and hence

$$\int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^2 \leq Cr^n$$

for some tame constant  $C > 0$ .

Therefore, by Hölder inequality,

$$\int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}| \leq \sqrt{\int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^2} \leq C,$$

up to renaming constants, as desired.  $\square$

Exploiting Theorem 1.1, we can now prove Corollary 1.2, by arguing as follows:

*Proof of Corollary 1.2.* By Theorem 1.1, we have that  $\nabla u \in L_{\text{loc}}^q(\Omega)$ , for any  $1 < q < +\infty$ . Hence  $u$  is continuous. The modulus of continuity  $\sigma$  follows as in [17] and [3]. Therefore,  $\Omega^+(u)$  is open and thus, if  $x_0 \in \Omega^+(u)$ , there exists  $r > 0$  such that  $\overline{B_r(x_0)} \subset \Omega^+(u)$ . Consequently,

$$m := \min_{\overline{B_r(x_0)}} u > 0.$$

Let now  $\phi \in C_0^\infty(B_r(x_0))$ ,  $\varepsilon \in \mathbb{R}$  and  $u_\varepsilon := u + \varepsilon\phi$ . If  $|\varepsilon| < \frac{m}{1+\|\phi\|_{L^\infty(\mathbb{R}^n)}}$ , we have that  $u_\varepsilon > 0$  in  $B_r(x_0)$  and so

$$\Omega^+(u_\varepsilon) = \Omega^+(u).$$

This implies that, for small  $\varepsilon$ ,

$$0 \leq J[u_\varepsilon] - J[u] = \int_{\Omega} (|\nabla u_\varepsilon|^2 - |\nabla u|^2)$$

and therefore

$$\int_{\Omega} \nabla u \cdot \nabla \phi = 0,$$

which shows that  $u$  is harmonic in  $B_r(x_0)$ , as desired.

Now we prove (1.17). For this we observe that, by the continuity of  $u$ , it follows that

$$\lim_{r \rightarrow 0} \int_{\partial B_r(x_0)} u = 0 \quad \text{for any } x_0 \in \Gamma.$$

Therefore

$$\begin{aligned}
(5.11) \quad \frac{1}{r^{n-1}} \int_{\partial B_r(x_0)} u &= \int_0^r \frac{d}{dt} \left( \frac{1}{t^{n-1}} \int_{\partial B_t(x_0)} u(x) d\mathcal{H}^{n-1}(x) \right) dt \\
&= \int_0^r \frac{d}{dt} \left( \int_{\partial B_t} u(x_0 + t\omega) d\mathcal{H}^{n-1}(\omega) \right) dt \\
&= \int_0^r \left( \int_{\partial B_t} \nabla u(x_0 + t\omega) \cdot \omega d\mathcal{H}^{n-1}(\omega) \right) dt \\
&= \int_0^r \left( \frac{1}{t^n} \int_{\partial B_t} \nabla u(x_0 + \nu) \cdot \nu d\mathcal{H}^{n-1}(\nu) \right) dt \\
&= \int_0^r \left( \frac{1}{t^n} \frac{d}{dt} \left( \int_{B_t} \nabla u(x_0 + \nu) \cdot \nu d\mathcal{H}^{n-1}(\nu) \right) \right) dt \\
&= \int_0^r \left( \frac{1}{t^{n-1}} \frac{d}{dt} \left( \int_{B_t(x_0)} \nabla u(x) \cdot \frac{x - x_0}{|x - x_0|} d\mathcal{H}^{n-1}(x) \right) \right) dt.
\end{aligned}$$

Now, we notice that

$$(5.12) \quad \int_{B_\varepsilon(x_0)} (\nabla u)_{x_0, \varepsilon} \cdot \frac{x - x_0}{|x - x_0|} dx = 0$$

by odd symmetry, for any  $\varepsilon > 0$ . In consequence of this, we have that

$$\begin{aligned}
\left| \frac{1}{\varepsilon^{n-1}} \int_{B_\varepsilon(x_0)} \nabla u(x) \cdot \frac{x - x_0}{|x - x_0|} dx \right| &= \left| \frac{1}{\varepsilon^{n-1}} \int_{B_\varepsilon(x_0)} (\nabla u(x) - (\nabla u)_{x_0, \varepsilon}) \cdot \frac{x - x_0}{|x - x_0|} dx \right| \\
&\leq \varepsilon \left( \frac{1}{\varepsilon^n} \int_{B_\varepsilon(x_0)} |\nabla u(x) - (\nabla u)_{x_0, \varepsilon}| dx \right) \rightarrow 0,
\end{aligned}$$

as  $\varepsilon \rightarrow 0$ , thanks to the BMO estimate in Theorem 1.1.

Thus, an integration by parts gives that

$$\begin{aligned}
&\int_0^r \left( \frac{1}{t^{n-1}} \frac{d}{dt} \left( \int_{B_t(x_0)} \nabla u(x) \cdot \frac{x - x_0}{|x - x_0|} d\mathcal{H}^{n-1}(x) \right) \right) dt \\
&= \frac{1}{r^{n-1}} \int_{B_r(x_0)} \nabla u(x) \cdot \frac{x - x_0}{|x - x_0|} d\mathcal{H}^{n-1}(x) + (n-1) \int_0^r \frac{1}{t^n} \int_{B_t(x_0)} \nabla u(x) \cdot \frac{x - x_0}{|x - x_0|} d\mathcal{H}^{n-1}(x) dt.
\end{aligned}$$

So, recalling (5.11) and using again (5.12), we obtain that

$$\begin{aligned}
\frac{1}{r^{n-1}} \int_{\partial B_r(x_0)} u &= \frac{1}{r^{n-1}} \int_{B_r(x_0)} (\nabla u(x) - (\nabla u)_{x_0, r}) \cdot \frac{x - x_0}{|x - x_0|} d\mathcal{H}^{n-1}(x) \\
&\quad + (n-1) \int_0^r \frac{1}{t^n} \int_{B_t(x_0)} (\nabla u(x) - (\nabla u)_{x_0, t}) \cdot \frac{x - x_0}{|x - x_0|} d\mathcal{H}^{n-1}(x) dt.
\end{aligned}$$

Therefore, the BMO estimate in Theorem 1.1 yields the desired result in (1.17).  $\square$

As customary, one can deduce from the integral estimate in (1.17) a linear growth from the free boundary. We give the details for convenience, starting from the one-phase case:

**Corollary 5.2.** *Let  $u \geq 0$  be a minimizer in  $\Omega$  for the functional  $J$  in (1.8) and let  $D \Subset \Omega$ . Let  $\varpi > 0$  and assume that  $\mathcal{L}^n(\Omega^+(u)) \geq \varpi$ .*

*Then, there exists  $C > 0$ , possibly depending on  $\varpi$ ,  $Q$ ,  $\Omega$  and  $D$ , such that*

$$u(x) \leq C \operatorname{dist}(x, \Gamma),$$

*for any  $x \in D$  for which  $B_{2\operatorname{dist}(x, \Gamma)}(x) \subset D$ .*

*Proof.* Let  $d$  be the distance of  $x$  to  $\Gamma$ . Let  $x_0 \in \overline{B_d(x)} \cap \Gamma$ . Then, we can use (1.17) and obtain that, for any  $\rho \in (0, 2d)$

$$\int_{\partial B_\rho(x_0)} u \leq C\rho^n,$$

for some  $C > 0$ . So, we integrate this inequality in  $\rho \in (0, 2d)$  and we find that

$$(5.13) \quad \int_{B_{2d}(x_0)} u \leq Cd^{n+1},$$

up to renaming  $C > 0$ .

On the other hand, since  $u$  is harmonic in  $B_d(x)$ , thanks to Corollary 1.2, we have that

$$(5.14) \quad u(x) = \oint_{B_d(x)} u.$$

Notice now that  $B_d(x) \subseteq B_{2d}(x_0)$ , hence we deduce from (5.13) and (5.14) that  $u(x) \leq Cd$ , as desired.  $\square$

From Corollary 5.2 one can deduce that  $u$  is Lipschitz continuous, as stated in the next result for completeness:

**Corollary 5.3.** *Let  $u \geq 0$  be a minimizer in  $\Omega$  for the functional  $J$  in (1.8) and let  $D \Subset \Omega$ . Let  $\varpi > 0$  and assume that  $\mathcal{L}^n(\Omega^+(u)) \geq \varpi$ .*

*Then, there exists  $C > 0$ , possibly depending on  $\varpi$ ,  $Q$ ,  $\Omega$  and  $D$ , such that*

$$\sup_{x \in D} |\nabla u(x)| \leq C.$$

The proof of Corollary 5.3 is by now standard (see e.g. Theorem 5.3 in [3]).

The Lipschitz estimate in Corollary 5.3 is optimal, since the solutions have linear growth from the free boundary, as stated in the following result:

**Lemma 5.4.** *Let  $u$  be a minimizer in  $\Omega$  for the functional  $J$  in (1.8) and let  $D \Subset \Omega$ .*

*Let  $d > 0$  and suppose that  $B_d \subseteq \Omega^+(u)$ .*

*Let  $\bar{\omega} \geq \varpi > 0$  and assume that*

$$(5.15) \quad \mathcal{L}^n(\Omega^+(u)) \in [\varpi, \bar{\omega}].$$

*Assume also that*

$$(5.16) \quad \Upsilon := \inf_{r \in [\lambda_2 Q_1 \varpi/2, 2\lambda_2 Q_2 \bar{\omega}]} \Phi'_0(r) > 0.$$

*Then, there exist  $d_0, c_0 > 0$ , possibly depending on  $\varpi, \bar{\omega}, Q, \lambda_1, \lambda_2$  and  $\Omega$ , such that if  $d \in (0, d_0)$  we have that*

$$u(0) \geq c_0 \Upsilon d.$$

*Proof.* We let

$$C_0 := \lambda_2 \int_{\Omega \setminus B_{d/2}} Q \chi_{\{u>0\}}.$$

We take  $d_0 > 0$  small enough such that

$$C_0 \geq \lambda_2 Q_1 \frac{\varpi}{2}.$$

In addition, we have that

$$C_0 \leq \lambda_2 Q_2 \bar{\omega}.$$

Then, for any  $a, b \in [0, \lambda_2 Q_2 \bar{\omega}]$ , with  $a \geq b$ , we have that

$$(5.17) \quad \Phi_0(C_0 + a) - \Phi_0(C_0 + b) \geq \Upsilon (a - b),$$

thanks to (5.16).

Also, from Corollary 1.2, we know that  $u$  is harmonic in  $B_d$  and so, by Harnack inequality,

$$(5.18) \quad \sup_{B_{d/2}} u \leq \bar{C} \inf_{B_{d/2}} u \leq \bar{C} u(0),$$

for some  $\bar{C} > 0$ .

We take  $\psi_0 \in C^\infty(\mathbb{R}^n)$ , such that  $\psi_0 = 0$  in  $B_{1/4}$ ,  $\psi_0 = 1$  on  $\partial B_{1/2}$  and  $|\nabla \psi_0| \leq 10$ . We also set

$$\psi(x) := 2\bar{C} u(0) \psi_0\left(\frac{x}{d}\right).$$

Notice that

$$(5.19) \quad |\nabla \psi| \leq \frac{20\bar{C} u(0)}{d}.$$

If  $x \in B_{d/2}$ , we define

$$v(x) := \min\{u(x), \psi(x)\}.$$

Notice that if  $x \in \partial B_{d/2}$ , then

$$\psi(x) = 2\bar{C} u(0) \geq u(x),$$

thanks to (5.18), hence  $v = u$  on  $\partial B_{d/2}$ . Therefore, we extend  $v(x) := u(x)$  for any  $x$  outside  $B_{d/2}$ , and we have that

$$(5.20) \quad \begin{aligned} 0 \leq J[v] - J[u] &= \int_{\Omega} |\nabla v|^2 - \int_{\Omega} |\nabla u|^2 + \Phi_0(\mathcal{M}_2(v)) - \Phi_0(\mathcal{M}_2(u)) \\ &= \int_{B_{d/2} \cap \{\psi > u\}} |\nabla \psi|^2 - \int_{B_{d/2} \cap \{\psi > u\}} |\nabla u|^2 + \Phi_0(\mathcal{M}_2(v)) - \Phi_0(\mathcal{M}_2(u)). \end{aligned}$$

Now, from (5.19), we have that

$$(5.21) \quad \int_{B_{d/2} \cap \{\psi > u\}} |\nabla \psi|^2 - \int_{B_{d/2} \cap \{\psi > u\}} |\nabla u|^2 \leq \int_{B_{d/2}} |\nabla \psi|^2 \leq \frac{400\bar{C}^2 \mathcal{L}^n(B_{d/2}) u^2(0)}{d^2}.$$

On the other hand

$$\begin{aligned} \Phi_0(\mathcal{M}_2(u)) - \Phi_0(\mathcal{M}_2(v)) &= \Phi_0\left(\lambda_2 \int_{\Omega} Q \chi_{\{u>0\}}\right) - \Phi_0\left(\lambda_2 \int_{\Omega} Q \chi_{\{v>0\}}\right) \\ &= \Phi_0\left(C_0 + \lambda_2 \int_{B_{d/2}} Q \chi_{\{u>0\}}\right) - \Phi_0\left(C_0 + \lambda_2 \int_{B_{d/2}} Q \chi_{\{v>0\}}\right). \end{aligned}$$

Notice that the quantity

$$\lambda_2 \int_{B_{d/2}} Q \chi_{\{u>0\}} + \lambda_2 \int_{B_{d/2}} Q \chi_{\{v>0\}}$$

is small if  $d$  is small enough, and so we can apply (5.17) with  $a := C_0 + \lambda_2 \int_{B_{d/2}} Q \chi_{\{u>0\}}$  and  $b := C_0 + \lambda_2 \int_{B_{d/2}} Q \chi_{\{v>0\}}$ . In this way, we find that

$$\begin{aligned} \Phi_0(\mathcal{M}_2(u)) - \Phi_0(\mathcal{M}_2(v)) &\geq \Upsilon \lambda_2 \int_{B_{d/2}} Q (\chi_{\{u>0\}} - \chi_{\{v>0\}}) \\ &= \lambda_2 \int_{B_{d/2} \cap \{u>\psi\}} Q (\chi_{\{u>0\}} - \chi_{\{\psi>0\}}). \end{aligned}$$

Notice also that in  $B_{d/4}$  we have that  $\psi = 0 < u$ , hence we conclude that

$$\Phi_0(\mathcal{M}_2(u)) - \Phi_0(\mathcal{M}_2(v)) \geq \lambda_2 \int_{B_{d/4}} Q (\chi_{\{u>0\}} - \chi_{\{\psi>0\}}) \geq \lambda_2 Q_1 \mathcal{L}^n(B_{d/4}).$$

Now, plugging this and (5.21) into (5.20) we infer

$$\frac{400\bar{C}^2 \mathcal{L}^n(B_{d/2}) u^2(0)}{d^2} \geq \lambda_2 Q_1 \mathcal{L}^n(B_{d/4}),$$

which implies the desired result.  $\square$

The Lipschitz regularity for the pure two-phase problem, as stated in Theorem 1.3, can be deduced from the BMO estimate, giving coherent growth for  $u^+$  and  $u^-$ , and the classical Alt-Caffarelli-Friedman monotonicity formula. The details go as follows:

*Proof of Theorem 1.3.* First we observe that, from (1.17), it follows that

$$(5.22) \quad \left| \frac{1}{r^{n-1}} \int_{\partial B_r(x)} u^+ - \frac{1}{r^{n-1}} \int_{\partial B_r(x)} u^- \right| \leq C,$$

for any  $x \in \Gamma$  and  $r > 0$  such that  $B_r(x) \subseteq D \Subset \Omega$ .

We recall now the Alt-Caffarelli-Friedman monotonicity formula [3]: Let  $w^+, w^-$  be two continuous, nonnegative subharmonic functions in  $B_1$ , with  $w_1 w_2 = 0$ ,  $w_1(0) = w_2(0) = 0$ . Then, for any  $x_0 \in \Gamma$ ,

$$\Phi(r, w_1, w_2) := \frac{1}{r^4} \int_{B_r(x_0)} \frac{|\nabla w_1(x)|^2}{|x - x_0|^{n-2}} dx \int_{B_r(x_0)} \frac{|\nabla w_2(x)|^2}{|x - x_0|^{n-2}} dx$$

is a monotone increasing function of  $r \in (0, 1)$ , and

$$(5.23) \quad \Phi(1, w_1, w_2) \leq C \left( 1 + \int_{B_1} w_1^2 + \int_{B_1} w_2^2 \right),$$

with some universal constant  $C > 0$ .

In what follows, we will apply this theorem with  $w_1 := u^+, w_2 := u^-$ .

To fix the ideas we assume that  $B_1 \subset D$ . Moreover, we take  $x_0 \in D$  such that  $u(x_0) > 0$  and let  $x \in \Gamma = \partial\{u > 0\}$  be the closest point to  $x_0$ , that is  $\text{dist}(x_0, \Gamma) = |x_0 - x|$ .

Setting  $\rho := |x - x_0|$ , we suppose that  $u(x_0) \geq M\rho > 0$ , for some large  $M > 0$ . Hence, applying the Harnack's inequality, we infer that

$$(5.24) \quad u \geq c_0 M \rho \quad \text{in } B_{\frac{3\rho}{4}}(x_0) \subset D,$$

for some  $c_0 > 0$ . Therefore, setting also

$$\Sigma_\rho := \partial B_\rho(x) \cap B_{\frac{3\rho}{4}}(x_0),$$

we conclude that

$$\oint_{\partial B_\rho(x)} u^+ \geq c_1 \oint_{\Sigma_\rho} u^+ \geq c_0 c_1 M \rho,$$

where  $c_1 > 0$  depends only on the dimension  $n$ .

From this and (5.22), we obtain that

$$(5.25) \quad \oint_{\partial B_\rho(x)} u^- \geq \oint_{\partial B_\rho(x)} u^+ - C\rho \geq (c_0 c_1 M - C)\rho > \frac{M}{2}\rho$$

if  $M$  is large enough.

Let  $y \in \partial B_{\frac{\rho}{2}}(x_0) \cap [x, x_0]$  be the mid-point of the segment  $[x, x_0]$ . Then, by construction, we have that

$$(5.26) \quad u^+ \geq c_0 M \rho \quad \text{in } B_{\frac{\rho}{4}}(y),$$

where  $c_0$  is the constant in (5.24).

For our next computation, it is convenient to switch to polar coordinates  $(r, \sigma)$  centered at  $x$ . Let  $E_\rho$  be the set of  $\sigma \in S^{n-1}$  such that  $u(x + \rho\sigma) < 0$ . Let also  $I_\sigma$  be the ray that connects  $y$  and  $x + \rho\sigma$ . In what follows, we parameterize  $I_\sigma$  in arc-length by the parameter  $r \geq 0$ , with  $r = 0$  corresponding to the point  $y$ . The function  $u$  evaluated at the point of  $I_\sigma$  parameterized by  $r$  will be denoted by  $u(r)$ .

In this notation, formula (5.26) says that  $u^-(r) = 0$  for any  $r \in (0, \frac{\rho}{4})$ , and so

$$\nabla u^-(r) = 0 \quad \text{for any } r \in \left(0, \frac{\rho}{4}\right).$$

Then, recalling (5.25), we have that

$$\begin{aligned}
(5.27) \quad \frac{M}{2} &\leq \frac{1}{\rho} \int_{\partial B_\rho(x)} u^- = \frac{C}{\rho} \int_{E_\rho} u^-(x + \rho\sigma) d\mathcal{H}^{n-1}(\sigma) = \frac{C}{\rho} \int_{E_\rho} \int_{I_\sigma \cap B_\rho(x)} D_r u^-(r) dr d\mathcal{H}^{n-1}(\sigma) \\
&\leq \frac{C}{\rho} (\rho \mathcal{H}^{n-1}(E_\rho))^{\frac{1}{2}} \left( \int_{E_\rho} \int_{I_\sigma \cap B_\rho(x)} |D_r u^-(r)|^2 dr d\mathcal{H}^{n-1}(\sigma) \right)^{\frac{1}{2}} \\
&\leq \frac{C}{\rho} (\rho \mathcal{H}^{n-1}(E_\rho))^{\frac{1}{2}} \left( \int_{S^{n-1}} \int_{I_\sigma \cap \{r \in [\rho/4, 2\rho]\}} |D_r u^-(r)|^2 dr d\mathcal{H}^{n-1}(\sigma) \right)^{\frac{1}{2}} \\
&\leq \frac{C}{\rho} (\rho \mathcal{H}^{n-1}(E_\rho))^{\frac{1}{2}} \left( \int_{B_{2\rho}(y) \setminus B_{\rho/4}(y)} \frac{|\nabla u^-(z)|^2}{|z - y|^{n-1}} dz \right)^{\frac{1}{2}} \\
&\leq \frac{C}{\rho} (\rho \mathcal{H}^{n-1}(E_\rho))^{\frac{1}{2}} \left( \frac{1}{\rho} \int_{B_{2\rho}(y) \setminus B_{\rho/4}(y)} \frac{|\nabla u^-(z)|^2}{|z - y|^{n-2}} dz \right)^{\frac{1}{2}},
\end{aligned}$$

up to renaming  $C > 0$  from line to line, where the Hölder's inequality was also used.

Now we observe that, if  $z \in B_{2\rho}(y) \setminus B_{\rho/4}(y)$ , we have that

$$|z - x| \leq |z - y| + |x - y| \leq 3\rho$$

and so

$$|z - y| \geq \frac{\rho}{4} \geq \frac{|z - x|}{12}.$$

Thus, renaming constants in (5.27), we obtain that

$$\begin{aligned}
(5.28) \quad \frac{M}{2} &\leq \frac{C}{\rho} (\rho \mathcal{H}^{n-1}(E_\rho))^{\frac{1}{2}} \left( \frac{1}{\rho} \int_{B_{3\rho}(x)} \frac{|\nabla u^-(z)|^2}{|z - x|^{n-2}} dz \right)^{\frac{1}{2}} \\
&= \frac{C}{\rho} (\mathcal{H}^{n-1}(E_\rho))^{\frac{1}{2}} \left( \int_{B_{3\rho}(x)} \frac{|\nabla u^-(z)|^2}{|z - x|^{n-2}} dz \right)^{\frac{1}{2}}.
\end{aligned}$$

In order to estimate the integral average of  $u^+$ , we use (5.26) and we obtain that (up to renaming constants)

$$\begin{aligned}
M\rho^n &\leq C \int_{\partial B_{\frac{\rho}{4}}(y)} u^+ = C\rho^{n-1} \int_{\partial B_1} u^+ \left(y + \frac{\rho}{4}\omega\right) d\mathcal{H}^{n-1}(\omega) \\
&= C\rho^{n-1} \int_{\partial B_1} \left[u^+ \left(y + \frac{\rho}{4}\omega\right) - u^+(x)\right] d\mathcal{H}^{n-1}(\omega) \\
&\leq C\rho^{n-1} \int_{\partial B_1} \left[\int_0^1 \left|\nabla u^+ \left(x + r \left(y - x + \frac{\rho}{4}\omega\right)\right) \cdot \left(y - x + \frac{\rho}{4}\omega\right)\right| dr\right] d\mathcal{H}^{n-1}(\omega) \\
&\leq C\rho^n \int_{\partial B_1} \left[\int_0^1 \left|\nabla u^+ \left(x + r \left(y - x + \frac{\rho}{4}\omega\right)\right)\right| dr\right] d\mathcal{H}^{n-1}(\omega) \\
&\leq C\rho^{n-1} \int_{\partial B_1} \left[\int_{\rho/4}^{2\rho} \left|\nabla u^+ (x + \bar{r}\bar{\omega})\right| d\bar{r}\right] d\mathcal{H}^{n-1}(\bar{\omega}) \\
&= C\rho^{n-1} \int_{\partial B_1} \left[\int_{\rho/4}^{2\rho} \frac{\bar{r}^{n-1} |\nabla u^+ (x + \bar{r}\bar{\omega})|}{\bar{r}^{n-1}} d\bar{r}\right] d\mathcal{H}^{n-1}(\bar{\omega}) \\
&= C\rho^{n-1} \int_{B_{2\rho}(x) \setminus B_{\rho/4}(x)} \frac{|\nabla u^+(z)|}{|z-x|^{n-1}} dz \\
&\leq C\rho^{n-1+\frac{n}{2}} \left(\int_{B_{2\rho}(x) \setminus B_{\rho/4}(x)} \frac{|\nabla u^+(z)|^2}{|z-x|^{2(n-1)}} dz\right)^{\frac{1}{2}} \\
&\leq C\rho^{n-1} \left(\int_{B_{2\rho}(x) \setminus B_{\rho/4}(x)} \frac{|\nabla u^+(z)|^2}{|z-x|^{n-2}} dz\right)^{\frac{1}{2}}.
\end{aligned}$$

Combining this with (5.28), and renaming constants, we conclude that

$$\begin{aligned}
M^2 &\leq \frac{C}{\rho^2} (\mathcal{H}^{n-1}(E_\rho))^{\frac{1}{2}} \left(\int_{B_{3\rho}(x)} \frac{|\nabla u^-(z)|^2}{|z-x|^{n-2}} dz\right)^{\frac{1}{2}} \left(\int_{B_{2\rho}(x) \setminus B_{\rho/4}(x)} \frac{|\nabla u^+(z)|^2}{|z-x|^{n-2}} dz\right)^{\frac{1}{2}} \\
&\leq \frac{C}{\rho} \left(\int_{B_{3\rho}(x)} \frac{|\nabla u^-(z)|^2}{|z-x|^{n-2}} dz\right)^{\frac{1}{2}} \left(\int_{B_{3\rho}(x)} \frac{|\nabla u^+(z)|^2}{|z-x|^{n-2}} dz\right)^{\frac{1}{2}} \\
&= C(\Phi(\rho, u^+, u^-))^{\frac{1}{2}}.
\end{aligned}$$

Hence, from the monotonicity formula and (5.23), we get that

$$M^2 \leq C \left(1 + \int_{B_1(x)} (u^+)^2 + \int_{B_1(x)} (u^-)^2\right)^{\frac{1}{2}},$$

which bounds  $M$ , as desired.  $\square$

## 6. FREE BOUNDARY CONDITION

In this section, we will assume that the function  $Q$  introduced in (1.4) is continuous.

Next result shows that, at points  $p$  of the free boundary, the following condition holds true in the sense of distributions:

$$\begin{aligned}
&(\partial_\nu^+ u(p))^2 - (\partial_\nu^- u(p))^2 \\
&= \left[ \lambda_2 \partial_{r_2} \Phi \left( \lambda_1 \int_\Omega Q \chi_{\{u < 0\}}, \lambda_2 \int_\Omega Q \chi_{\{u > 0\}} \right) - \lambda_1 \partial_{r_1} \Phi \left( \lambda_1 \int_\Omega Q \chi_{\{u < 0\}}, \lambda_2 \int_\Omega Q \chi_{\{u > 0\}} \right) \right] Q(p),
\end{aligned}$$

where  $\nu$  is the normal vector exterior to  $\partial\{u > 0\}$  (and thus pointing towards  $\{u \leq 0\}$ ) and we set

$$(6.1) \quad \partial_\nu^+ u(x) := \lim_{t \rightarrow 0} \frac{u(x - t\nu) - u(x)}{t} \quad \text{and} \quad \partial_\nu^- u(x) := \lim_{t \rightarrow 0} \frac{u(x + t\nu) - u(x)}{t}.$$

More precisely, we have that:

**Lemma 6.1.** *Let  $u$  be a minimizer of  $J$  as in (1.8) and suppose that  $Q \in W^{1,1}(\Omega)$ . Assume also that*

$$(6.2) \quad \text{the set } \{u = 0\} \cap \Omega \text{ has zero measure.}$$

Then

$$\lim_{\varepsilon \searrow 0} \left\{ \int_{\partial\{u > \varepsilon\} \cap \Omega} \mathcal{I}^{\varepsilon,+}(u, x) V(x) \cdot \nu^{\varepsilon,+}(x) d\mathcal{H}^{n-1}(x) + \int_{\partial\{u < -\varepsilon\} \cap \Omega} \mathcal{I}^{\varepsilon,-}(u, x) V(x) \cdot \nu^{\varepsilon,-}(x) d\mathcal{H}^{n-1}(x) \right\} = 0$$

for any vector field  $V \in C_0^\infty(\Omega, \mathbb{R}^n)$ , where we have denoted by  $\nu^{\varepsilon,+}$  and  $\nu^{\varepsilon,-}$  the exterior normals of  $\{u > \varepsilon\}$  and  $\{u < -\varepsilon\}$ , respectively, and

$$(6.3) \quad \begin{aligned} \mathcal{I}^{\varepsilon,+}(u, x) &:= |\partial_\nu^+ u(x)|^2 - \lambda_2 \partial_{r_2} \Phi \left( \lambda_1 \int_\Omega Q(\xi) \chi_{\{u < -\varepsilon\}}(\xi) d\xi, \lambda_2 \int_\Omega Q(\xi) \chi_{\{u > \varepsilon\}}(\xi) d\xi \right) Q(x) \\ \text{and } \mathcal{I}^{\varepsilon,-}(u, x) &:= |\partial_\nu^- u(x)|^2 - \lambda_1 \partial_{r_1} \Phi \left( \lambda_1 \int_\Omega Q(\xi) \chi_{\{u < -\varepsilon\}}(\xi) d\xi, \lambda_2 \int_\Omega Q(\xi) \chi_{\{u > \varepsilon\}}(\xi) d\xi \right) Q(x). \end{aligned}$$

*Proof.* The argument is a (not completely straightforward) modification of the classical domain variations in Theorem 2.5 in [2] and in Theorem 2.4 of [3]. We provide full details for the facility of the reader. For small  $t \in \mathbb{R}$ , we consider the ODE flow  $y = y(t; x)$  given by the Cauchy problem

$$\begin{cases} \partial_t y(t; x) = V(y(t; x)), \\ y(0; x) = x. \end{cases}$$

The map  $\mathbb{R}^n \ni x \mapsto y(t; x)$  is invertible for small  $t$ , i.e. we can consider the inverse diffeomorphism  $x(t; y)$  and we define

$$u_t(y) := u(x(t; y)).$$

We remark that, in light of (6.2),

$$(6.4) \quad \text{the set } \{u_t = 0\} \cap \Omega \text{ has zero measure.}$$

Given  $\varepsilon > 0$ , we define  $E^{\varepsilon,+} := \{u > \varepsilon\} \cap \Omega$ ,  $E^{\varepsilon,-} := \{u < -\varepsilon\} \cap \Omega$  and  $E_t^{\varepsilon,\pm} := y(t; E^{\varepsilon,\pm})$ . Notice that

$$(6.5) \quad \{u_t > \varepsilon\} \cap \Omega = E_t^{\varepsilon,+} \quad \text{and} \quad \{u_t < -\varepsilon\} \cap \Omega = E_t^{\varepsilon,-}.$$

One can check (see e.g. formulas (4.5), (4.13) and (4.22) in [12]) that

$$(6.6) \quad y(t; \Omega) = \Omega,$$

$$(6.7) \quad \det D_x y(t; x) = 1 + t \operatorname{div} V(x) + o(t)$$

$$(6.8) \quad \text{and} \quad \begin{aligned} \int_\Omega |\nabla u(x)|^2 dx - \int_\Omega |\nabla u_t(y)|^2 dy &= \lim_{\varepsilon \searrow 0} t \int_{\partial E^{\varepsilon,+} \cap \Omega} |\partial_\nu^+ u(y)|^2 V(y) \cdot \nu^{\varepsilon,+}(y) d\mathcal{H}^{n-1}(y) \\ &\quad + t \int_{\partial E^{\varepsilon,-} \cap \Omega} |\partial_\nu^- u(y)|^2 V(y) \cdot \nu^{\varepsilon,-}(y) d\mathcal{H}^{n-1}(y) + o(t). \end{aligned}$$

By approximating  $Q$  by a sequence of smooth functions in  $W^{1,1}(\Omega)$  and a.e. in  $\Omega$ , we deduce from (6.6) and (6.7) that

$$\begin{aligned} \int_\Omega Q(y) \chi_{E_t^{\varepsilon,\pm}}(y) dy &= \int_\Omega Q(y(t; x)) \chi_{E^{\varepsilon,\pm}}(x) (1 + t \operatorname{div} V(x) + o(t)) dx \\ &= \int_\Omega Q(x) \chi_{E^{\varepsilon,\pm}}(x) (1 + t \operatorname{div} V(x) + o(t)) dx \\ &\quad + \int_{\Omega \times (0,1)} \nabla Q(x + \tau(y(t; x) - x)) \cdot (y(t; x) - x) \chi_{E^{\varepsilon,\pm}}(x) (1 + t \operatorname{div} V(x) + o(t)) dx d\tau. \end{aligned}$$



Therefore

$$\begin{aligned}
 (6.9) \quad & \int_{\Omega} Q(y) \chi_{E_t^{\varepsilon, \pm}}(y) dy \\
 &= \int_{\Omega} Q(x) \chi_{E^{\varepsilon, \pm}}(x) dx + t \int_{\Omega} \operatorname{div}(Q(x)V(x)) \chi_{E^{\varepsilon, \pm}}(x) dx - t \int_{E^{\varepsilon, \pm}} \nabla Q(x) \cdot V(x) dx \\
 &\quad + t \int_{E^{\varepsilon, \pm} \times (0,1)} \nabla Q(x + \tau t V(x) + o(t)) \cdot V(x) dx d\tau + o(t).
 \end{aligned}$$

Furthermore, changing variables again, we see that

$$\begin{aligned}
 (6.10) \quad & \int_{E^{\varepsilon, \pm} \times (0,1)} \nabla Q(x + \tau t V(x) + o(t)) \cdot V(x) dx d\tau \\
 &= \int_{\tilde{E}_t^{\varepsilon, \pm} \times (0,1)} \nabla Q(y) \cdot V(y - \tau t V(y) + o(t)) (1 - t \operatorname{div} V(y) + o(t)) dy d\tau \\
 &= \int_{\tilde{E}_t^{\varepsilon, \pm} \times (0,1)} \nabla Q(y) \cdot V(y) dy d\tau + o(1) \\
 &= \int_{E^{\varepsilon, \pm}} \nabla Q(y) \cdot V(y) dy + o(1),
 \end{aligned}$$

for a suitable set  $\tilde{E}_t^{\varepsilon, \pm}$  which is close, for small  $t$ , to  $E^{\varepsilon, \pm}$ . From (6.9) and (6.10), we have that

$$\int_{\Omega} Q(y) \chi_{E_t^{\varepsilon, \pm}}(y) dy = \int_{\Omega} (Q(x) + t \operatorname{div}(Q(x)V(x))) \chi_{E^{\varepsilon, \pm}}(x) dx + o(t).$$

Consequently, we can linearize  $\Phi$  and obtain

$$\begin{aligned}
 (6.11) \quad & \Phi \left( \lambda_1 \int_{\Omega} Q(y) \chi_{E_t^{\varepsilon, -}}(y) dy, \lambda_2 \int_{\Omega} Q(y) \chi_{E_t^{\varepsilon, +}}(y) dy \right) \\
 &= \Phi \left( \lambda_1 \int_{\Omega} Q(x) \chi_{E^{\varepsilon, -}}(x) dx, \lambda_2 \int_{\Omega} Q(x) \chi_{E^{\varepsilon, +}}(x) dx \right) \\
 &\quad + t \lambda_1 \partial_{r_1} \Phi \left( \lambda_1 \int_{\Omega} Q(x) \chi_{E^{\varepsilon, -}}(x) dx, \lambda_2 \int_{\Omega} Q(x) \chi_{E^{\varepsilon, +}}(x) dx \right) \int_{\Omega} \operatorname{div}(Q(x)V(x)) \chi_{E^{\varepsilon, -}}(x) dx \\
 &\quad + t \lambda_2 \partial_{r_2} \Phi \left( \lambda_1 \int_{\Omega} Q(x) \chi_{E^{\varepsilon, -}}(x) dx, \lambda_2 \int_{\Omega} Q(x) \chi_{E^{\varepsilon, +}}(x) dx \right) \int_{\Omega} \operatorname{div}(Q(x)V(x)) \chi_{E^{\varepsilon, +}}(x) dx + o(t) \\
 &= \Phi \left( \lambda_1 \int_{\Omega} Q(x) \chi_{E^{\varepsilon, -}}(x) dx, \lambda_2 \int_{\Omega} Q(x) \chi_{E^{\varepsilon, +}}(x) dx \right) \\
 &\quad + t \lambda_1 \partial_{r_1} \Phi \left( \lambda_1 \int_{\Omega} Q(x) \chi_{E^{\varepsilon, -}}(x) dx, \lambda_2 \int_{\Omega} Q(x) \chi_{E^{\varepsilon, +}}(x) dx \right) \int_{\partial E^{\varepsilon, -} \cap \Omega} Q(x)V(x) \cdot \nu^{\varepsilon, -}(x) d\mathcal{H}^{n-1}(x) \\
 &\quad + t \lambda_2 \partial_{r_2} \Phi \left( \lambda_1 \int_{\Omega} Q(x) \chi_{E^{\varepsilon, -}}(x) dx, \lambda_2 \int_{\Omega} Q(x) \chi_{E^{\varepsilon, +}}(x) dx \right) \int_{\partial E^{\varepsilon, +} \cap \Omega} Q(x)V(x) \cdot \nu^{\varepsilon, +}(x) d\mathcal{H}^{n-1}(x) \\
 &\quad + o(t).
 \end{aligned}$$

Moreover, by inspection and recalling (6.5), one sees that

$$\lim_{\varepsilon \searrow 0} \chi_{E_t^{\varepsilon, +}} = \lim_{\varepsilon \searrow 0} \chi_{\{u_t > \varepsilon\}} = \chi_{\{u_t > 0\}},$$

for any small  $t \geq 0$ . Similarly, and using (6.4),

$$\lim_{\varepsilon \searrow 0} \chi_{E_t^{\varepsilon, -}} = \chi_{\{u_t < 0\}} = \chi_{\{u_t \leq 0\}}$$

a.e. in  $\Omega$ . As a consequence, by the Dominated Convergence Theorem,

$$\mathcal{M}_1(u_t) = \lim_{\varepsilon \searrow 0} \lambda_1 \int_{\Omega} Q(x) \chi_{E_t^{\varepsilon, -}}(x) dx \quad \text{and} \quad \mathcal{M}_2(u_t) = \lim_{\varepsilon \searrow 0} \lambda_2 \int_{\Omega} Q(x) \chi_{E_t^{\varepsilon, +}}(x) dx,$$

for any small  $t \geq 0$ . So, we can take the limit with respect to  $\varepsilon$  in formula (6.11) and obtain that

$$\begin{aligned} & \Phi(\mathcal{M}_1(u_t), \mathcal{M}_2(u_t)) - \Phi(\mathcal{M}_1(u), \mathcal{M}_2(u)) \\ &= \lim_{\varepsilon \searrow 0} t \lambda_1 \partial_{r_1} \Phi \left( \lambda_1 \int_{\Omega} Q(x) \chi_{E^{\varepsilon,-}}(x) dx, \lambda_2 \int_{\Omega} Q(x) \chi_{E^{\varepsilon,+}}(x) dx \right) \int_{\partial E^{\varepsilon,-} \cap \Omega} Q(x) V(x) \cdot \nu^{\varepsilon,-}(x) d\mathcal{H}^{n-1}(x) \\ & \quad + t \lambda_2 \partial_{r_2} \Phi \left( \lambda_1 \int_{\Omega} Q(x) \chi_{E^{\varepsilon,-}}(x) dx, \lambda_2 \int_{\Omega} Q(x) \chi_{E^{\varepsilon,+}}(x) dx \right) \int_{\partial E^{\varepsilon,+} \cap \Omega} Q(x) V(x) \cdot \nu^{\varepsilon,+}(x) d\mathcal{H}^{n-1}(x) \\ & \quad + o(t). \end{aligned}$$

From this and (6.8), we have that

$$\begin{aligned} & J[u] - J[u_t] \\ &= \int_{\Omega} |\nabla u(x)|^2 dx - \int_{\Omega} |\nabla u_t(y)|^2 dy \\ & \quad + \Phi(\mathcal{M}_1(u), \mathcal{M}_2(u)) - \Phi(\mathcal{M}_1(u_t), \mathcal{M}_2(u_t)) \\ &= \lim_{\varepsilon \searrow 0} t \int_{\partial E^{\varepsilon,+} \cap \Omega} |\partial_{\nu}^+ u(y)|^2 V(y) \cdot \nu^{\varepsilon,+}(y) d\mathcal{H}^{n-1}(y) + t \int_{\partial E^{\varepsilon,-} \cap \Omega} |\partial_{\nu}^- u(y)|^2 V(y) \cdot \nu^{\varepsilon,-}(y) d\mathcal{H}^{n-1}(y) \\ & \quad - t \lambda_1 \partial_{r_1} \Phi \left( \lambda_1 \int_{\Omega} Q(x) \chi_{E^{\varepsilon,-}}(x) dx, \lambda_2 \int_{\Omega} Q(x) \chi_{E^{\varepsilon,+}}(x) dx \right) \int_{\partial E^{\varepsilon,-} \cap \Omega} Q(x) V(x) \cdot \nu^{\varepsilon,-}(x) d\mathcal{H}^{n-1}(x) \\ & \quad - t \lambda_2 \partial_{r_2} \Phi \left( \lambda_1 \int_{\Omega} Q(x) \chi_{E^{\varepsilon,-}}(x) dx, \lambda_2 \int_{\Omega} Q(x) \chi_{E^{\varepsilon,+}}(x) dx \right) \int_{\partial E^{\varepsilon,+} \cap \Omega} Q(x) V(x) \cdot \nu^{\varepsilon,+}(x) d\mathcal{H}^{n-1}(x) \\ & \quad + o(t). \end{aligned}$$

Dividing by  $t$  and then letting  $t \rightarrow 0$ , we obtain the desired result.  $\square$

We observe that when  $\Phi(r_1, r_2) := r_1 + r_2$ , then (6.3) reduces to  $\mathcal{I}^{\varepsilon,+}(u, x) := |\partial_{\nu}^+ u(x)|^2 - \lambda_2 Q(x)$  and  $\mathcal{I}^{\varepsilon,-}(u, x) := |\partial_{\nu}^- u(x)|^2 - \lambda_1 Q(x)$ . Hence, in this particular case, our Lemma 6.1 boils down to Theorem 2.4 in [3]. If also  $\lambda_1 := 0$ , then Lemma 6.1 boils down to Theorem 2.5 in [2].

Next we show that  $\partial\{u > 0\}$  contains  $\partial\{u < 0\}$  under the condition (1.7). Thus one has sharp separation of phases.

**Lemma 6.2.** *Let  $u$  be a minimizer in  $\Omega$  of the functional in (1.8). Then  $\partial\{u < 0\} \setminus \partial\{u > 0\} = \emptyset$ .*

*Proof.* Let  $E := \partial\{u < 0\} \setminus \partial\{u > 0\}$ . We want to show that  $E$  is empty and suppose by contradiction that  $E \neq \emptyset$ . Then, there exist  $p \in \Omega$  and  $r > 0$  such that  $u \leq 0$  in  $B_r(p)$ , with  $u(p) = 0$  and  $\mathcal{L}^n(B_r(p) \cap \{u < 0\}) > 0$ . Then, we use that  $u$  is subharmonic, in view of Lemma 2.2, and we obtain that

$$u(p) \leq \int_{B_r(p)} u < 0,$$

which is a contradiction.  $\square$

## 7. NONDEGENERACY OF MINIMIZERS

One of the fundamental properties of the minimizers is a linear lower bound. In other words, the minimizers grow at least linearly away from the free boundary. This is the content of Theorem 1.4, that we now prove:

*Proof of Theorem 1.4.* Let  $\kappa \in (0, 1)$ . Without loss of generality we assume that  $x_0 = 0$ .

Notice that the critical points of a non-constant harmonic function have Hausdorff dimension less than  $n - 2$ . From Sard's theorem it follows that the one dimensional Lebesgue measure of the critical values of  $u$  is zero. Consequently,  $\partial\{u > \varepsilon\}$  is a regular surface for a.e.  $\varepsilon > 0$ . In particular, one can choose  $\varepsilon > 0$  small enough to ensure that  $B_{\kappa r} \cap \{u > \varepsilon\} \neq \emptyset$ .

Now take any such small  $\varepsilon > 0$ , and consider the problem

$$(7.1) \quad \begin{cases} v_\varepsilon = u & \text{on } \partial B_r, \\ v_\varepsilon = u & \text{in } B_r \cap \left( \{u < \varepsilon\} \cup (\{u > \varepsilon\} \setminus \Omega_0^+) \right), \\ v_\varepsilon = \varepsilon & \text{in } B_{\kappa r} \cap \{u > \varepsilon\} \cap \Omega_0^+, \\ \Delta v_\varepsilon = 0 & \text{in } D_\varepsilon^+, \end{cases}$$

where

$$(7.2) \quad D_\varepsilon^+ = (B_r \setminus B_{\kappa r}) \cap \{u > \varepsilon\} \cap \Omega_0^+.$$

Observe that  $v_\varepsilon$  can be obtained by minimizing the Dirichlet integral over  $B_r$  subject to the constraints in (7.1). In fact, the function

$$w_\varepsilon = \begin{cases} u & \text{in } \{u < \varepsilon\} \\ \varepsilon & \text{in } \{u > \varepsilon\} \cap \Omega_0^+ \cap B_{\kappa r} \\ u & \text{elsewhere} \end{cases}$$

satisfies the boundary constraints in (7.1), and hence

$$(7.3) \quad \int_{B_r} |\nabla v_\varepsilon|^2 \leq \int_{B_r} |\nabla w_\varepsilon|^2 \leq C,$$

for some tame constant  $C > 0$  independent of  $\varepsilon$ . Also,  $v_\varepsilon$  is continuous at  $\{u = \varepsilon\} \cap (B_r \setminus B_{\kappa r})$ . We claim that

$$(7.4) \quad v_\varepsilon \leq u \text{ in } \overline{D_\varepsilon^+}.$$

Indeed, by inspection we see that  $v_\varepsilon \leq u$  on  $\partial D_\varepsilon^+$ . Moreover,  $D_\varepsilon^+ \subseteq \{u > \varepsilon\}$  and so  $u$  is harmonic there, thanks to Corollary 1.2. Hence, we obtain (7.4) from the comparison principle.

Now, formulas (7.3) and (7.4) imply that  $v_\varepsilon \rightarrow v$  weakly in  $W^{1,2}(B_r)$ , as  $\varepsilon \rightarrow 0$ , and  $v \leq u$ . Furthermore,  $v$  is continuous in  $B_r$  and solves

$$\begin{cases} v = u & \text{on } \partial B_r, \\ v = u & \text{in } B_r \cap \left( \{u \leq 0\} \cup (\{u > 0\} \setminus \Omega_0^+) \right), \\ v = 0 & \text{in } B_{\kappa r} \cap \Omega_0^+, \\ \Delta v = 0 & \text{in } D^+, \end{cases}$$

where

$$(7.5) \quad D^+ := (B_r \setminus B_{\kappa r}) \cap \Omega_0^+.$$

The former follows from a customary approximation argument as on page 437 of [3], and hence omitted here.

Now we extend  $v$  to be equal to  $u$  in  $\Omega \setminus B_r$  and we compare  $J[u]$  with  $J[v]$  in  $\Omega$ . Accordingly, the minimality of  $u$  gives that

$$\int_{\Omega} |\nabla u|^2 + \Phi_0 \left( \lambda_2 \int_{\Omega} Q \chi_{\{u > 0\}} \right) \leq \int_{\Omega} |\nabla v|^2 + \Phi_0 \left( \lambda_2 \int_{\Omega} Q \chi_{\{v > 0\}} \right).$$

This implies that

$$(7.6) \quad \int_{B_r} |\nabla u|^2 - \int_{B_r} |\nabla v|^2 = \int_{\Omega} |\nabla u|^2 - \int_{\Omega} |\nabla v|^2 \leq \Phi_0 \left( \lambda_2 \int_{\Omega} Q \chi_{\{v > 0\}} \right) - \Phi_0 \left( \lambda_2 \int_{\Omega} Q \chi_{\{u > 0\}} \right).$$

On the other hand, recalling that  $v = u$  in  $B_r \setminus \Omega_0^+$  and  $v = 0$  in  $B_{\kappa r} \cap \Omega_0^+$ , we see that

$$(7.7) \quad \begin{aligned} \int_{B_r} |\nabla u|^2 - |\nabla v|^2 &= \int_{B_r \cap \Omega_0^+} |\nabla u|^2 - |\nabla v|^2 \\ &= \int_{D^+} |\nabla u|^2 - |\nabla v|^2 + \int_{B_{\kappa r} \cap \Omega_0^+} |\nabla u|^2, \end{aligned}$$

where (7.5) was also used. This and (7.6) give that

$$(7.8) \quad \int_{D^+} |\nabla u|^2 - |\nabla v|^2 + \int_{B_{\kappa r} \cap \Omega_0^+} |\nabla u|^2 \leq \Phi_0 \left( \lambda_2 \int_{\Omega} Q\chi_{\{v>0\}} \right) - \Phi_0 \left( \lambda_2 \int_{\Omega} Q\chi_{\{u>0\}} \right).$$

Now we write

$$(7.9) \quad \begin{aligned} & \Phi_0 \left( \lambda_2 \int_{\Omega} Q\chi_{\{u>0\}} \right) - \Phi_0 \left( \lambda_2 \int_{\Omega} Q\chi_{\{v>0\}} \right) \\ &= \Phi_0 \left( \lambda_2 \int_{\Omega \setminus B_{\kappa r}} Q\chi_{\{u>0\}} + \lambda_2 \int_{B_{\kappa r}} Q\chi_{\{u>0\}} \right) - \Phi_0 \left( \lambda_2 \int_{\Omega \setminus B_{\kappa r}} Q\chi_{\{v>0\}} + \lambda_2 \int_{B_{\kappa r}} Q\chi_{\{v>0\}} \right). \end{aligned}$$

Notice that  $v = 0$  in  $B_{\kappa r} \cap \Omega_0^+$  and  $v = u$  in  $B_{\kappa r} \setminus \Omega_0^+$ , hence

$$(7.10) \quad \int_{B_{\kappa r}} Q\chi_{\{v>0\}} = \int_{B_{\kappa r} \setminus \Omega_0^+} Q\chi_{\{u>0\}}.$$

Moreover,  $v = u$  in  $\Omega \setminus B_r$  and in  $(B_r \setminus B_{\kappa r}) \setminus \Omega_0^+$ . Also, in  $(B_r \setminus B_{\kappa r}) \cap \Omega_0^+ = D^+$ , we have that  $u > 0$ , by definition of  $\Omega_0^+$ , and therefore

$$\chi_{\{u>0\}} \geq \chi_{\{v>0\}} \quad \text{in } D^+.$$

This implies that

$$(7.11) \quad \int_{\Omega \setminus B_{\kappa r}} Q\chi_{\{v>0\}} \leq \int_{\Omega \setminus B_{\kappa r}} Q\chi_{\{u>0\}}.$$

Plugging (7.10) and (7.11) into (7.9), and using (1.7), we obtain that

$$\begin{aligned} & \Phi_0 \left( \lambda_2 \int_{\Omega} Q\chi_{\{u>0\}} \right) - \Phi_0 \left( \lambda_2 \int_{\Omega} Q\chi_{\{v>0\}} \right) \\ & \geq \Phi_0 \left( \lambda_2 \int_{\Omega \setminus B_{\kappa r}} Q\chi_{\{u>0\}} + \lambda_2 \int_{B_{\kappa r}} Q\chi_{\{u>0\}} \right) - \Phi_0 \left( \lambda_2 \int_{\Omega \setminus B_{\kappa r}} Q\chi_{\{u>0\}} + \lambda_2 \int_{B_{\kappa r} \setminus \Omega_0^+} Q\chi_{\{u>0\}} \right). \end{aligned}$$

Therefore, from the Mean Value Theorem we get

$$(7.12) \quad \begin{aligned} & \Phi_0 \left( \lambda_2 \int_{\Omega} Q\chi_{\{u>0\}} \right) - \Phi_0 \left( \lambda_2 \int_{\Omega} Q\chi_{\{v>0\}} \right) \\ & \geq \Theta \left( \lambda_2 \int_{B_{\kappa r}} Q\chi_{\{u>0\}} - \lambda_2 \int_{B_{\kappa r} \setminus \Omega_0^+} Q\chi_{\{u>0\}} \right) = \Theta \lambda_2 \int_{B_{\kappa r} \cap \Omega_0^+} Q\chi_{\{u>0\}}, \end{aligned}$$

where we recalled the notation in (1.19), and (1.18) has been used to estimate the interval in the definition of  $\Theta$ .

So, from (7.8) and (7.12) we deduce that

$$\int_{D^+} |\nabla u|^2 - |\nabla v|^2 + \int_{B_{\kappa r} \cap \Omega_0^+} |\nabla u|^2 \leq -\Theta \lambda_2 \int_{B_{\kappa r} \cap \Omega_0^+} Q\chi_{\{u>0\}}.$$

Using this, we obtain that

$$(7.13) \quad \min \{1, \Theta\} \int_{B_{\kappa r} \cap \Omega_0^+} (|\nabla u|^2 + \lambda_2 Q\chi_{\{u>0\}}) \leq \int_{D^+} |\nabla v|^2 - |\nabla u|^2.$$

Now we observe that

$$\begin{aligned} \int_{D^+} |\nabla v|^2 - |\nabla u|^2 &= \int_{D^+} (\nabla v - \nabla u) \cdot (\nabla u - \nabla v + 2\nabla v) \\ &= - \int_{D^+} |\nabla u - \nabla v|^2 + 2 \int_{D^+} \nabla v \cdot (\nabla v - \nabla u) \leq 2 \int_{D^+} \nabla v \cdot (\nabla v - \nabla u). \end{aligned}$$

Plugging this into (7.13), recalling that  $v_\varepsilon$  is a solution of (7.1) and using the Divergence Theorem, we obtain

$$\begin{aligned}
(7.14) \quad & \min\{1, \Theta\} \int_{B_{\kappa r} \cap \Omega_0^+} (|\nabla u|^2 + \lambda_2 Q \chi_{\{u>0\}}) \\
& \leq 2 \int_{D^+} \nabla v \cdot (\nabla v - \nabla u) \\
& \leq 2 \liminf_{\varepsilon \rightarrow 0} \int_{D_\varepsilon^+} \nabla v_\varepsilon \cdot (\nabla v_\varepsilon - \nabla u) \\
& = 2 \liminf_{\varepsilon \rightarrow 0} \left[ \int_{D_\varepsilon^+} \operatorname{div}(\nabla v_\varepsilon (v_\varepsilon - u)) - \int_{D_\varepsilon^+} \Delta v_\varepsilon (v_\varepsilon - u) \right] \\
& = 2 \liminf_{\varepsilon \rightarrow 0} \int_{\partial D_\varepsilon^+} \frac{\partial v_\varepsilon}{\partial \nu} (v_\varepsilon - u) \\
& \leq 2 \liminf_{\varepsilon \rightarrow 0} \int_{\partial B_{\kappa r} \cap \{u>\varepsilon\} \cap \Omega_0^+} \left| \frac{\partial v_\varepsilon}{\partial \nu} \right| (u - \varepsilon).
\end{aligned}$$

Notice that we have performed an integration by parts, which actually needs a justification, since  $D_\varepsilon^+$  has not smooth boundary. This can be done using an approximation of  $D_\varepsilon^+$  by domains whose boundaries are  $C^\infty$  curves, as in [3] (see in particular page 437 there).

Our next goal is to estimate the quantity

$$2 \liminf_{\varepsilon \rightarrow 0} \int_{\partial B_{\kappa r} \cap \{u>\varepsilon\} \cap \Omega_0^+} \left| \frac{\partial v_\varepsilon}{\partial \nu} \right| (u - \varepsilon).$$

For this, we set  $\kappa' := (\kappa + 1)/2$  and we introduce the barrier  $b$  as follows:

$$(7.15) \quad \begin{cases} b = \varepsilon + \sup_{\partial B_{\kappa' r} \cap \Omega_0^+} v_\varepsilon & \text{on } \partial B_{\kappa' r}, \\ b = \varepsilon & \text{on } \partial B_{\kappa r}, \\ \Delta b = 0 & \text{in } B_{\kappa' r} \setminus B_{\kappa r}. \end{cases}$$

Notice that  $b \geq \varepsilon$  on  $\partial(B_{\kappa' r} \setminus B_{\kappa r})$  and so, by comparison principle, we have that

$$(7.16) \quad b \geq \varepsilon \text{ in } \overline{B_{\kappa' r}} \setminus B_{\kappa r}.$$

Recalling (7.2), we set

$$D_{\varepsilon,*}^+ := D_\varepsilon^+ \cap B_{\kappa' r} = (B_{\kappa' r} \setminus B_{\kappa r}) \cap \{u > \varepsilon\} \cap \Omega_0^+$$

and we claim that

$$(7.17) \quad v_\varepsilon \leq b \text{ on } \partial D_{\varepsilon,*}^+.$$

For this, we use the elementary formula, given sets  $A$  and  $B$ ,

$$(7.18) \quad \partial(A \cap B) \subseteq (\partial A \cap \overline{B}) \cup (\partial B \cap \overline{A}).$$

This gives that

$$\partial D_{\varepsilon,*}^+ \subseteq D_1 \cup D_2,$$

with

$$\begin{aligned}
D_1 &:= (\partial B_{\kappa r} \cup \partial B_{\kappa' r}) \cap \{x \in \overline{\Omega_0^+} \text{ s.t. } u(x) \geq \varepsilon\} \cap \overline{D_\varepsilon^+} \\
\text{and } D_2 &:= (\overline{B_{\kappa' r}} \setminus B_{\kappa r}) \cap (\partial\{x \in \Omega_0^+ \text{ s.t. } u(x) > \varepsilon\}) \cap \overline{D_\varepsilon^+}.
\end{aligned}$$

Now, in light of (7.15) and (7.1), if  $x \in D_1$ , we have that either  $x \in \partial B_{\kappa' r} \cap \{u \geq \varepsilon\} \cap \overline{\Omega_0^+}$ , and then  $b(x) \geq \sup_{\partial B_{\kappa' r} \cap \Omega_0^+} v_\varepsilon \geq v_\varepsilon(x)$ , or  $x \in \partial B_{\kappa r} \cap \{u \geq \varepsilon\} \cap \overline{\Omega_0^+}$  and  $b(x) = \varepsilon = v_\varepsilon(x)$ . Accordingly,

$$(7.19) \quad b \geq v_\varepsilon \text{ in } D_1.$$

On the other hand, using again (7.18), one sees that

$$\partial\{x \in \Omega_0^+ \text{ s.t. } u(x) > \varepsilon\} \subseteq (\partial\Omega_0^+) \cup (\partial\{u > \varepsilon\}) \subseteq (\partial\Omega) \cup (\partial\{u > 0\}) \cup (\partial\{u > \varepsilon\})$$

and so

$$D_2 \subseteq (\overline{B_{\kappa'r}} \setminus B_{\kappa r}) \cap (\{u = 0\} \cup \{u = \varepsilon\}) \cap \overline{D_\varepsilon^+}.$$

As a consequence, if  $x \in D_2$ , then  $x \in \overline{B_{\kappa'r}} \setminus B_{\kappa r}$  and  $u(x) \leq \varepsilon$ . This and (7.16) give that  $u \leq b$  in  $D_2$ . Hence, in view of (7.4), we find that  $v_\varepsilon \leq b$  in  $D_2$ . This, together with (7.19), proves (7.17).

Now, by (7.1), (7.15), (7.17) and the comparison principle, we conclude that  $v_\varepsilon \leq b$  in  $D_{\varepsilon,*}^+$ . Therefore, since  $v_\varepsilon = \varepsilon = b$  on  $\partial B_{\kappa r} \cap \{u > \varepsilon\} \cap \Omega_0^+$ , we find that

$$(7.20) \quad |\nabla v_\varepsilon| \leq |\nabla b| \text{ on } \partial B_{\kappa r} \cap \{u > \varepsilon\} \cap \Omega_0^+.$$

As a matter of fact, we can explicitly solve  $b$  in (7.15), and we have that

$$b(x) = \frac{\sup_{\partial B_{\kappa'r} \cap \Omega_0^+} v_\varepsilon}{\Psi(\kappa r) - \Psi(\kappa'r)} (\Psi(\kappa'r) - \Psi(|x|)) + \varepsilon + \sup_{\partial B_{\kappa'r} \cap \Omega_0^+} v_\varepsilon,$$

where  $\Psi$  is the radially decreasing fundamental solution of the Laplace operator in  $\mathbb{R}^n$  (up to normalizing constants,  $\Psi(\rho) = \rho^{2-n}$  if  $n \geq 3$  and  $\Psi(\rho) = -\log \rho$  if  $n = 2$ ).

Consequently, recalling also (7.4),

$$(7.21) \quad \sup_{\partial B_{\kappa r}} |\nabla b| = C \frac{\sup_{\partial B_{\kappa'r} \cap \Omega_0^+} v_\varepsilon}{\Psi(\kappa r) - \Psi(\kappa'r)} |\nabla \Psi(\kappa r)| \leq C \frac{\sup_{\partial B_{\kappa'r} \cap \Omega_0^+} u}{r},$$

for some  $C > 0$  possibly depending on  $\kappa$  and different from step to step.

Now, we observe that, extending  $u$  by zero outside  $\Omega_0^+$ , we obtain a nonnegative subharmonic function. More precisely, if we set  $\tilde{u} := u\chi_{\Omega_0^+}$ , given any nonnegative  $\phi \in C_0^\infty(\Omega)$ , we have that

$$\begin{aligned} \int_{\Omega} \tilde{u} \Delta \phi &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega_0^+ \cap \{u > \varepsilon\}} u \Delta \phi = \lim_{\varepsilon \rightarrow 0} \int_{\Omega_0^+ \cap \{u > \varepsilon\}} (\phi \Delta u + \operatorname{div}(u \nabla \phi - \phi \nabla u)) \\ &= 0 + \lim_{\varepsilon \rightarrow 0} \int_{\Omega_0^+ \cap \partial \{u > \varepsilon\}} u \frac{\partial \phi}{\partial \nu} - \phi \frac{\partial u}{\partial \nu} = - \lim_{\varepsilon \rightarrow 0} \int_{\Omega_0^+ \cap \partial \{u > \varepsilon\}} \phi \frac{\partial u}{\partial \nu} \geq 0, \end{aligned}$$

and so  $\tilde{u}$  is subharmonic. Hence the weak maximum principle can be applied to the function  $\tilde{u}$ , and so we conclude that, for any  $\sigma \in (0, 1)$  and any  $x \in B_{\sigma r}$ ,

$$\begin{aligned} u(x)\chi_{\Omega_0^+}(x) = \tilde{u}(x) &\leq \int_{B_{(1-\sigma)r}(x)} \tilde{u} \leq \left( \int_{B_{(1-\sigma)r}(x)} \tilde{u}^2 \right)^{\frac{1}{2}} \leq \left( \frac{1}{\mathcal{L}^n(B_{(1-\sigma)r})} \int_{B_r} \tilde{u}^2 \right)^{\frac{1}{2}} \\ &\leq \left( \frac{\mathcal{L}^n(B_r \cap \Omega_0^+)}{\mathcal{L}^n(B_{(1-\sigma)r})} \int_{B_r \cap \Omega_0^+} u^2 \right)^{\frac{1}{2}} \leq \left( \frac{r^n}{(1-\sigma)^n r^n} \int_{B_r \cap \Omega_0^+} u^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Thus,

$$(7.22) \quad \sup_{B_{\sigma r} \cap \Omega_0^+} u \leq \frac{1}{(1-\sigma)^{\frac{n}{2}}} \left( \int_{B_r \cap \Omega_0^+} u^2 \right)^{\frac{1}{2}} = \frac{r\gamma}{(1-\sigma)^{\frac{n}{2}}},$$

where we have set

$$\gamma := \left( \frac{1}{r^2} \int_{B_r \cap \Omega_0^+} u^2 \right)^{\frac{1}{2}}.$$

From this and (7.21) we conclude that

$$(7.23) \quad \sup_{\partial B_{\kappa r}} |\nabla b| \leq C\gamma.$$

From (7.20) and (7.23), it follows that

$$\begin{aligned} \int_{\partial B_{\kappa r} \cap \{u > \varepsilon\} \cap \Omega_0^+} \left| \frac{\partial v_\varepsilon}{\partial \nu} \right| (u - \varepsilon) &\leq \int_{\partial B_{\kappa r} \cap \{u > \varepsilon\} \cap \Omega_0^+} |\nabla v_\varepsilon| (u - \varepsilon) \\ &\leq \int_{\partial B_{\kappa r} \cap \{u > \varepsilon\} \cap \Omega_0^+} |\nabla b| (u - \varepsilon) \leq C\gamma \int_{\partial B_{\kappa r} \cap \{u > \varepsilon\} \cap \Omega_0^+} (u - \varepsilon) \\ &\leq C\gamma \int_{\partial B_{\kappa r} \cap \Omega_0^+} u. \end{aligned}$$

So, making use of (7.14), we obtain

$$(7.24) \quad \int_{B_{\kappa r} \cap \Omega_0^+} (|\nabla u|^2 + \lambda_2 Q \chi_{\{u > 0\}}) \leq C\gamma \int_{\partial B_{\kappa r} \cap \Omega_0^+} u.$$

Now we recall the elementary trace inequality for nonnegative functions  $f$ , see e.g. Theorem 1(ii) on page 258 of [13], namely

$$(7.25) \quad \int_{\partial B_\rho} f \leq C \left[ \int_{B_\rho} |\nabla f| + \frac{1}{\rho} \int_{B_\rho} f \right].$$

We apply (7.25) to  $f := u \chi_{\Omega_0^+}$  and  $\rho := \kappa r$ . Then, since  $u$  vanishes along  $\partial \Omega_0^+$ ,

$$\int_{\partial B_{\kappa r} \cap \Omega_0^+} u \leq C \left[ \int_{B_{\kappa r} \cap \Omega_0^+} |\nabla u| + \frac{1}{r} \int_{B_{\kappa r} \cap \Omega_0^+} u \right],$$

where  $C > 0$  now may also depend on  $\kappa$ .

Hence, in light of (7.24), we find that

$$(7.26) \quad \int_{B_{\kappa r} \cap \Omega_0^+} (|\nabla u|^2 + \lambda_2 Q \chi_{\{u > 0\}}) \leq C\gamma \left[ \int_{B_{\kappa r} \cap \Omega_0^+} |\nabla u| + \frac{1}{r} \int_{B_{\kappa r} \cap \Omega_0^+} u \right].$$

Now we point out that, by the Cauchy-Schwarz inequality,

$$2 \int_{B_{\kappa r} \cap \Omega_0^+} |\nabla u| \leq \int_{B_{\kappa r} \cap \Omega_0^+} (|\nabla u|^2 + 1).$$

This and (7.26) give that

$$\begin{aligned} &\int_{B_{\kappa r} \cap \Omega_0^+} (|\nabla u|^2 + \lambda_2 Q \chi_{\{u > 0\}}) \\ &\leq C\gamma \int_{B_{\kappa r} \cap \Omega_0^+} (|\nabla u|^2 + \chi_{\{u > 0\}}) + \frac{C\gamma}{r} \sup_{B_{\kappa r} \cap \Omega_0^+} u \int_{B_{\kappa r} \cap \Omega_0^+} \chi_{\{u > 0\}} \\ &\leq C\gamma \max \left\{ 1, \frac{1}{Q_1 \lambda_2} \right\} \left[ \int_{B_{\kappa r} \cap \Omega_0^+} (|\nabla u|^2 + \lambda_2 Q \chi_{\{u > 0\}}) + \frac{1}{r} \sup_{B_{\kappa r} \cap \Omega_0^+} u \int_{B_{\kappa r} \cap \Omega_0^+} Q \chi_{\{u > 0\}} \right]. \end{aligned}$$

In consequence of this and (7.22), we obtain that

$$\int_{B_{\kappa r} \cap \Omega_0^+} (|\nabla u|^2 + \lambda_2 Q \chi_{\{u > 0\}}) \leq C\gamma(1 + \gamma) \int_{B_{\kappa r} \cap \Omega_0^+} (|\nabla u|^2 + \lambda_2 Q \chi_{\{u > 0\}}).$$

If  $\gamma$  is sufficiently small we conclude that  $u$  vanishes identically in  $B_{\kappa r} \cap \Omega_0^+$ , as desired.  $\square$

## 8. DENSITY THEOREMS AND CLEAN BALL CONDITIONS

In this section we prove that the positive phase  $\{u > 0\}$  occupies a positive density near the free boundary points.

**Theorem 8.1.** *Assume that  $u$  is a minimizer of  $J$  as in (1.8). Let  $\varpi > 0$  and assume that  $\mathcal{L}^n(\Omega^+(u)) \geq \varpi$ . Let  $D \Subset \Omega$ . Assume that  $x_0 \in \Gamma = \partial\Omega^+(u)$  and let  $r > 0$  be such that  $B_r(x_0) \subseteq D$ .*

*Then, there exist  $c_1 \in (0, 1)$ , possibly depending on  $\varpi$ ,  $Q$ ,  $\Omega$  and  $D$ , and  $y_0 \in B_r(x_0)$  such that*

$$(8.1) \quad B_{c_1 r}(y_0) \subseteq B_r(x_0) \cap \Omega^+(u).$$

*Moreover, there exists  $c_2 > 0$ , possibly depending on  $\varpi$ ,  $Q$ ,  $\Omega$  and  $D$ , such that*

$$(8.2) \quad \mathcal{L}^n(B_r(x_0) \cap \Omega^+(u)) \geq c_2 r^n.$$

*Proof.* Obviously, we have that (8.2) is a direct consequence of (8.1), so we focus on the proof of (8.1). To this aim, we recall that  $u$  is continuous, thanks to Corollary 1.2, hence we can take  $y_0 \in \overline{B_{r/2}(x_0)}$  such that

$$(8.3) \quad u(y_0) = \max_{\overline{B_{r/2}(x_0)}} u.$$

We take  $d := \text{dist}(y_0, \Gamma)$  and  $z_0 \in \Gamma \cap \partial B_d(y_0)$ . Notice that, since  $x_0 \in \Gamma$ , we have that

$$(8.4) \quad d \leq |x_0 - y_0| \leq \frac{r}{2}.$$

Hence, we are in the position of applying Corollary 5.2, and we obtain that

$$(8.5) \quad u(y_0) \leq Cd,$$

for some  $C > 0$ . On the other hand, from Theorem 1.4 and (8.3),

$$cr^2 \leq \int_{B_{r/2}(x_0)} u^2 \leq u^2(y_0),$$

for some  $c > 0$ . Comparing this with (8.5), we conclude that

$$d \geq \frac{u(y_0)}{C} \geq \frac{\sqrt{c}r}{C} = c_1 r,$$

for some  $c_1 > 0$ . As a matter of fact, by (8.4), we have that  $c_1 \in (0, 1/2)$ . This construction establishes (8.1).  $\square$

## 9. BLOW-UP LIMITS

In this section, we consider the blow-up of a minimizer at a free boundary point. We will show that, in the limit, we obtain a minimizer for the Alt-Caffarelli problem in (1.11). This phenomenon plays an important role in our analysis, since it transforms the original nonlinear free boundary problem into a linear one, in the blow-up limit: that is, in our framework, the blow-up possesses an additional linearization feature.

To this extent, for any  $x_0 \in \Gamma$  we consider the blow-up sequence of  $u$  at  $x_0$ , that is

$$(9.1) \quad u_k(x) := \frac{u(x_0 + \rho_k x)}{\rho_k},$$

where  $\rho_k \rightarrow 0$  as  $k \rightarrow +\infty$ .

We have the following convergence result (see e.g. Proposition 8.1 in [11] for the proof):

**Proposition 9.1.** *Let  $x_0 \in \Gamma$  and  $u_k$  a the blow-up sequence, as introduced in (9.1).*

*Then there exists a blow-up limit  $u_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ , which is continuous and with linear growth, such that, up to a subsequence, as  $k \rightarrow +\infty$ ,*

- $u_k \rightarrow u_0$  in  $C_{loc}^\alpha(\mathbb{R}^n)$  for any  $\alpha \in (0, 1)$ ,
- $\nabla u_k \rightarrow \nabla u_0$  weakly in  $L_{loc}^q(\mathbb{R}^n)$  for any  $q > 1$ ,
- $\partial\{u_k > 0\} \rightarrow \partial\{u_0 > 0\}$  locally in Hausdorff distance,
- $\chi_{\{u_k > 0\}} \rightarrow \chi_{\{u_0 > 0\}}$  in  $L_{loc}^1(\mathbb{R}^n)$ .

We remark that in the proof of Proposition 9.1 we do not need Lebesgue density estimates. The statement above can be also enhanced, giving the pointwise convergence of the gradients, as given by the next result:



**Lemma 9.2.** *Let  $x_0 \in \Gamma$ . Let  $u_k$  be the blow-up sequence, as introduced in (9.1), and  $u_0$  the blow-up limit given by Proposition 9.1. Then  $\nabla u_k \rightarrow \nabla u_0$  a.e. in  $\mathbb{R}^n$ , as  $k \rightarrow +\infty$ .*

*In addition, if  $p \in \{u_0 \neq 0\}$ , we have that  $\nabla u_k \rightarrow \nabla u_0$  as  $k \rightarrow +\infty$  uniformly in a neighborhood of  $p$ .*

*Proof.* The proof is an appropriate modification (and actually a simplification) of some arguments also exploited in [2]. We let  $\mathcal{A}$  be the set of the Lebesgue density points of  $\{u_0 = 0\}$ . We show that  $\nabla u_k \rightarrow \nabla u_0$  in  $\mathcal{A} \cup \{u_0 \neq 0\}$ , as  $k \rightarrow +\infty$ , with locally uniform convergence in  $\{u_0 \neq 0\}$  (with this, since the complement of  $\mathcal{A} \cup \{u_0 \neq 0\}$  has zero Lebesgue measure, the desired result is established).

To this aim, we observe that if  $p \in \{u_0 \neq 0\}$  we know from Proposition 9.1 that there exists  $r_0 > 0$  such that  $u_k(x) \neq 0$  for any  $x \in B_{r_0}(p)$ , as long as  $k$  is large enough. Then, by Corollary 1.2, we have that  $u_k$  is harmonic in  $B_{r_0}(p)$  and so it has second derivatives estimates in  $B_{r_0/2}(p)$ . This implies that  $\nabla u_k \rightarrow \nabla u_0$  uniformly in  $B_{r_0/2}(p)$  and so, in particular, that  $\nabla u_k(p) \rightarrow \nabla u_0(p)$ , as  $k \rightarrow +\infty$ .

Now, let us take  $q \in \mathcal{A}$ . Then,

$$\lim_{r \searrow 0} \frac{\mathcal{L}^n(B_r(q) \cap \{u_0 = 0\})}{\mathcal{L}^n(B_r(q))} = 1$$

and therefore for any  $\eta > 0$  there exists  $\bar{r}(\eta) > 0$  such that if  $r \in (0, \bar{r}(\eta)]$  then

$$\frac{\mathcal{L}^n(B_r(q) \cap \{u_0 = 0\})}{\mathcal{L}^n(B_r(q))} \geq 1 - \eta.$$

In particular  $\mathcal{L}^n(B_r(q) \cap \{u_0 \neq 0\}) \leq \eta \mathcal{L}^n(B_r(q))$  and so, in light of the Lipschitz regularity obtained in Theorem 1.3, we have that

$$\int_{B_r(q)} u_0^2 = \frac{1}{\mathcal{L}^n(B_r(q))} \int_{B_r(q) \cap \{u_0 \neq 0\}} u_0^2 \leq C r^2 \frac{\mathcal{L}^n(B_r(q) \cap \{u_0 \neq 0\})}{\mathcal{L}^n(B_r(q))} \leq C \eta r^2 < \frac{c}{2} r^2,$$

up to renaming  $C > 0$  and taking  $\eta$  suitably small, where  $c > 0$  is the one given in Theorem 1.4. Consequently

$$\int_{B_r(q)} u_k^2 < c r^2$$

if  $k$  is large enough, and so, by Theorem 1.4, we have that  $u_k \leq 0$  in  $B_r(q)$ , and so  $u_0 \leq 0$  in  $B_r(q)$ .

We also know that  $u_k$  is subharmonic, thanks to Lemma 2.2, and thus also  $u_0$  is subharmonic. Accordingly, for small  $r > 0$ ,

$$0 = u_0(q) \leq \int_{B_r(q)} u_0 \leq 0,$$

which implies that  $u_0$  vanishes identically in  $B_r(q)$ . Similarly,  $u_k$  vanishes identically in  $B_r(q)$ . These considerations imply that  $\nabla u_k(q) = 0 = \nabla u_0(q)$ .  $\square$

Next result shows that the blow-up limit  $u_0$  is always a minimizer of the Alt-Caffarelli functional in (1.11) (for a suitable choice of  $Q$ , which turns out to be constant). That is, the blow-up limit has the additional, and somehow unexpected advantage, to linearize the interfacial energy. The precise result, which was stated in Theorem 1.5, is proved by the following argument:

*Proof of Theorem 1.5.* The result in Theorem 1.5 can be seen as a nonlinear counterpart of Lemma 5.4 in [2]. Up to a translation, we take  $x_0 := 0$  in (9.1). We take a competitor  $v_0$  for  $u_0$ , i.e. we suppose that  $v_0 - u_0 \in W_0^{1,2}(B_r)$ . We also take  $\eta \in C_0^\infty(B_r, [0, 1])$  and we define

$$v_\rho := v_0 + (1 - \eta)(u_\rho - u_0).$$

We observe that

$$(9.2) \quad v_\rho - u_\rho = (v_0 - u_0) - \eta(u_\rho - u_0)$$

and so

$$(9.3) \quad v_\rho - u_\rho = 0 \text{ outside } B_r.$$

In addition,

$$\{v_\rho > 0\} \subseteq \{v_0 > 0\} \cup \{\eta < 1\}$$

and therefore

$$(9.4) \quad \chi_{\{v_\rho > 0\}} \leq \chi_{\{v_0 > 0\}} + \chi_{\{\eta < 1\}}.$$

We also define  $v(x) := \rho v_\rho(x/\rho)$ . We remark that

$$(v - u)(x) = \rho \left( v_\rho \left( \frac{x}{\rho} \right) - u_\rho \left( \frac{x}{\rho} \right) \right) = 0 \quad \text{for any } x \in \mathbb{R}^n \setminus B_{\rho r},$$

thanks to (9.1) and (9.3). Since  $B_{\rho r} \subset \Omega$  when  $\rho$  is sufficiently small (possibly in dependence of the fixed  $r > 0$ ), we obtain that  $v - u = 0$  outside  $\Omega$ .

Consequently, we can use the minimality of  $u$  in  $\Omega$  and find that

$$(9.5) \quad \begin{aligned} 0 &\leq J[v] - J[u] = \int_{\Omega} (|\nabla v|^2 - |\nabla u|^2) + \Phi_0 \left( \lambda_2 \int_{\Omega} Q \chi_{\{v > 0\}} \right) - \Phi_0 \left( \lambda_2 \int_{\Omega} Q \chi_{\{u > 0\}} \right) \\ &= \int_{B_{\rho r}} (|\nabla v|^2 - |\nabla u|^2) + \Phi_0 \left( \lambda_2 \int_{B_{\rho r}} Q \chi_{\{v > 0\}} + \Xi_\rho \right) - \Phi_0 \left( \lambda_2 \int_{B_{\rho r}} Q \chi_{\{u > 0\}} + \Xi_\rho \right), \end{aligned}$$

where

$$\Xi_\rho := \lambda_2 \int_{\Omega \setminus B_{\rho r}} Q \chi_{\{u > 0\}}.$$

We point out that

$$(9.6) \quad \lim_{\rho \searrow 0} \Xi_\rho = \Xi_0 := \lambda_2 \int_{\Omega} Q \chi_{\{u > 0\}}.$$

Now we scale the quantities in (9.5), using the substitution  $y := x/\rho$ . In this way, we find that

$$\begin{aligned} \int_{B_{\rho r}} |\nabla v(x)|^2 dx &= \int_{B_{\rho r}} |\nabla v_\rho(x/\rho)|^2 dx = \rho^n \int_{B_r} |\nabla v_\rho(y)|^2 dy \\ \text{and} \quad \int_{B_{\rho r}} Q(x) \chi_{\{v > 0\}}(x) dx &= \int_{B_{\rho r}} Q(x) \chi_{\{v_\rho > 0\}}(x/\rho) dx = \rho^n \int_{B_r} Q(\rho y) \chi_{\{v_\rho > 0\}}(y) dy, \end{aligned}$$

and similar expressions hold true with  $u$  replacing  $v$ . Substituting these identities into (9.5), we conclude that

$$(9.7) \quad \begin{aligned} 0 &\leq \int_{B_r} (|\nabla v_\rho|^2 - |\nabla u_\rho|^2) \\ &\quad + \frac{1}{\rho^n} \left[ \Phi_0 \left( \rho^n \lambda_2 \int_{B_r} Q(\rho x) \chi_{\{v_\rho > 0\}}(x) dx + \Xi_\rho \right) - \Phi_0 \left( \rho^n \lambda_2 \int_{B_r} Q(\rho x) \chi_{\{u_\rho > 0\}}(x) dx + \Xi_\rho \right) \right]. \end{aligned}$$

Now, from (9.2), we have that

$$v_\rho + u_\rho = v_0 - u_0 - \eta(u_\rho - u_0) + 2u_\rho = (v_0 + u_0) + (2 - \eta)(u_\rho - u_0).$$

This and (9.2) give that

$$\begin{aligned} |\nabla v_\rho|^2 - |\nabla u_\rho|^2 &= \nabla(v_\rho + u_\rho) \cdot (v_\rho - u_\rho) \\ &= \nabla((v_0 + u_0) + (2 - \eta)(u_\rho - u_0)) \cdot \nabla((v_0 - u_0) - \eta(u_\rho - u_0)) \\ &= \nabla(v_0 + u_0) \cdot \nabla(v_0 - u_0) - \eta \nabla(v_0 + u_0) \cdot \nabla(u_\rho - u_0) - (u_\rho - u_0) \nabla(v_0 + u_0) \cdot \nabla \eta \\ &\quad - (u_\rho - u_0) \nabla \eta \cdot \nabla(v_0 - u_0) + (2 - \eta) \nabla(u_\rho - u_0) \cdot \nabla(v_0 - u_0) \\ &\quad + |u_\rho - u_0|^2 |\nabla \eta|^2 + \eta(u_\rho - u_0) \nabla \eta \cdot \nabla(u_\rho - u_0) \\ &\quad - (2 - \eta)(u_\rho - u_0) \nabla(u_\rho - u_0) \cdot \nabla \eta - (2 - \eta) \eta |\nabla(u_\rho - u_0)|^2. \end{aligned}$$

We remark that the latter term has a sign. So, recalling Proposition 9.1, we obtain that

$$(9.8) \quad \lim_{\rho \searrow 0} \int_{B_r} (|\nabla v_\rho|^2 - |\nabla u_\rho|^2) \leq \int_{B_r} \nabla(v_0 + u_0) \cdot \nabla(v_0 - u_0) = \int_{B_r} (|\nabla v_0|^2 - |\nabla u_0|^2).$$

Now we set

$$\begin{aligned} \alpha_\rho &:= \lambda_2 \int_{B_r} Q(\rho x) \chi_{\{v_0 > 0\}}(x) dx + \lambda_2 \int_{B_r} Q(\rho x) \chi_{\{\eta < 1\}}(x) dx \\ \text{and} \quad \beta_\rho &:= \lambda_2 \int_{B_r} Q(\rho x) \chi_{\{u_\rho > 0\}}(x) dx. \end{aligned}$$

We observe that

$$\begin{aligned} \lim_{\rho \searrow 0} \alpha_\rho &= \alpha_0 := \lambda_2 \int_{B_r} Q(0) \chi_{\{v_0 > 0\}}(x) dx + \lambda_2 \int_{B_r} Q(0) \chi_{\{\eta < 1\}}(x) dx \\ \text{and} \quad \lim_{\rho \searrow 0} \beta_\rho &= \beta_0 := \lambda_2 \int_{B_r} Q(0) \chi_{\{u_0 > 0\}}(x) dx, \end{aligned}$$

thanks to Proposition 9.1.

Then, recalling (9.4) and the monotonicity of  $\Phi_0$ , and exploiting also (9.6), we have that

$$\begin{aligned} & \lim_{\rho \searrow 0} \frac{1}{\rho^n} \left[ \Phi_0 \left( \rho^n \lambda_2 \int_{B_r} Q(\rho x) \chi_{\{v_\rho > 0\}}(x) dx + \Xi_\rho \right) - \Phi_0 \left( \rho^n \lambda_2 \int_{B_r} Q(\rho x) \chi_{\{u_\rho > 0\}}(x) dx + \Xi_\rho \right) \right] \\ & \leq \lim_{\rho \searrow 0} \frac{1}{\rho^n} \left[ \Phi_0(\rho^n \alpha_\rho + \Xi_\rho) - \Phi_0(\rho^n \beta_\rho + \Xi_\rho) \right] \\ & = \lim_{\rho \searrow 0} (\alpha_\rho - \beta_\rho) \int_0^1 \Phi'_0(\rho^n \beta_\rho + t \rho^n (\alpha_\rho - \beta_\rho) + \Xi_\rho) dt \\ & = (\alpha_0 - \beta_0) \Phi'_0(\Xi_0). \end{aligned}$$

So, we insert this inequality and (9.8) into (9.7) and we obtain

$$\begin{aligned} 0 & \leq \int_{B_r} (|\nabla v_0|^2 - |\nabla u_0|^2) + (\alpha_0 - \beta_0) \Phi'_0(\Xi_0) \\ & = \int_{B_r} (|\nabla v_0|^2 - |\nabla u_0|^2) + \lambda_2 Q(0) \Phi'_0(\Xi_0) \int_{B_r} \chi_{\{v_0 > 0\}}(x) dx \\ & \quad + \lambda_2 Q(0) \Phi'_0(\Xi_0) \int_{B_r} \chi_{\{\eta < 1\}}(x) dx - \lambda_2 Q(0) \Phi'_0(\Xi_0) \int_{B_r} \chi_{\{u_0 > 0\}}(x) dx. \end{aligned}$$

This estimate is valid for any choice of the function  $\eta$ ; therefore, letting  $\{\eta = 1\}$  invade the whole of  $B_r$ , we deduce that the term  $\int_{B_r} \chi_{\{\eta < 1\}}(x) dx$  can be made as small as we wish. As a consequence,

$$0 \leq \int_{B_r} (|\nabla v_0|^2 - |\nabla u_0|^2) + \lambda_2 Q(0) \Phi'_0(\Xi_0) \int_{B_r} \chi_{\{v_0 > 0\}}(x) dx - \lambda_2 Q(0) \Phi'_0(\Xi_0) \int_{B_r} \chi_{\{u_0 > 0\}}(x) dx,$$

which establishes the desired minimality property for  $u_0$ .  $\square$

## 10. PARTIAL REGULARITY OF THE FREE BOUNDARY

Using the subharmonicity property of the minimizers and their Lipschitz regularity (recall Lemma 2.2 and Theorem 1.3), we are now in the position to exploit standard techniques from geometric measure theory and conclude a partial regularity of the free boundary, as claimed in Theorem 1.6:

*Proof of Theorem 1.6.* The proof of (i) in Theorem 1.6 follows by a standard integration by parts (combined with the subharmonicity property of Lemma 2.2, see e.g. the proof of formula (8.1) in [11] for details).

To prove (ii) in Theorem 1.6, we argue by contradiction and we suppose that there exist  $r_j \searrow 0$  and  $x_j \in \Gamma$  such that  $B_{2r_j}(x_j) \subset \Omega$  and

$$(10.1) \quad \int_{B_{r_j}(x_j)} \Delta u^+ \leq \frac{r_j^{n-1}}{j}.$$

Up to subsequences, we may suppose that  $x_j \rightarrow \bar{x} \in \overline{D} \subset \Omega$ . We define

$$v_j(x) := \frac{u(x_j + r_j x)}{r_j}.$$

Notice that, in the light of Corollary 5.3, for any  $x, y \in \overline{B_1}$  we have

$$|v_j(x) - v_j(y)| \leq \frac{|u(x_j + r_j x) - u(x_j + r_j y)|}{r_j} \leq \frac{C|(x_j + r_j x) - (x_j + r_j y)|}{r_j} \leq C|x - y|.$$

So  $v_j$  (and hence  $v_j^+$ ) is Lipschitz in  $\overline{B_1}$  uniformly in  $j$  and thus we may suppose that  $v_j^+$  converges to some  $\bar{v}^+$  uniformly in  $\overline{B_1}$ .

Now, let  $\phi \in C_0^\infty(B_1, [0, +\infty))$ . Notice that, by Lemma 2.2, we know that  $u$  is subharmonic, hence so is  $u^+$  and  $\Delta u^+ \geq 0$ . Hence, recalling (10.1), we have

$$\begin{aligned} \int_{B_1} \bar{v}^+ \Delta \phi &= \lim_{j \rightarrow +\infty} \int_{B_1} v_j^+ \Delta \phi = \lim_{j \rightarrow +\infty} r_j^{-1} \int_{B_1} u^+(x_j + r_j x) \Delta \phi(x) dx \\ &= \lim_{j \rightarrow +\infty} r_j \int_{B_1} \Delta u^+(x_j + r_j x) \phi(x) dx = \lim_{j \rightarrow +\infty} r_j^{1-n} \int_{B_{r_j}(x_j)} \Delta u^+(\xi) \phi\left(\frac{\xi - x_j}{r_j}\right) d\xi \\ &\leq \sup_{\mathbb{R}^n} \phi \lim_{j \rightarrow +\infty} r_j^{1-n} \int_{B_{r_j}(x_j)} \Delta u^+(\xi) d\xi \leq \sup_{\mathbb{R}^n} \phi \lim_{j \rightarrow +\infty} \frac{1}{j} = 0. \end{aligned}$$

Accordingly,  $\bar{v}^+$  is superharmonic in  $B_1$ . By construction, we also have that  $\bar{v}^+ \geq 0$  and

$$\bar{v}^+(0) = \lim_{j \rightarrow +\infty} \frac{u(x_j)}{r_j} = \lim_{j \rightarrow +\infty} \frac{0}{r_j} = 0.$$

As a consequence

$$(10.2) \quad \bar{v}^+ \text{ vanishes identically in } B_1.$$

On the other hand, by (1.20),

$$cr_j^2 \leq \int_{B_{r_j}(x_j) \cap \Omega_0^+} u^2 \leq \sup_{B_{r_j}(x_j)} u^2 \leq r_j^2 \sup_{B_1} v_j^2,$$

for some  $c > 0$ . So, simplifying  $r_j$  on both sides of the inequality and taking the limit in  $j$ , we find that

$$c \leq \sup_{B_1} \bar{v}^2.$$

This is in contradiction with (10.2) and so the proof of (ii) is complete.

Then, (1.22) follows from point (ii) and suitable geometric measure theory arguments (see e.g. the proof of Corollary 8.2 in [11] for full details). For the proof of (1.23), see e.g. the proof of Theorem B in [11].  $\square$

## 11. REGULARITY OF THE FREE BOUNDARY

In this section we show that at flat points the free boundary is a regular smooth surface. In particular, in two spatial dimensions the free boundary is a continuously differentiable curve.

Our approach differs from the one in [2, 3] as we avoid using the flatness classes. Instead, we use the free boundary regularity theory for the viscosity solutions from [11].

**11.1. Viscosity solutions.** Recall the definitions of  $\Omega^+(u)$  and  $\Omega^-(u)$  given in the section with the notation.

If the free boundary is  $C^1$  smooth, then

$$G(u_\nu^+, u_\nu^-) := (u_\nu^+)^2 - (u_\nu^-)^2 - \Lambda$$

is the flux balance across the free boundary, where  $u_\nu^+$  and  $u_\nu^-$  are the normal derivatives in the inward direction to  $\partial\Omega^+(u)$  and  $\partial\Omega^-(u)$ , respectively, and  $\Lambda$  is defined in (1.14).

With this notation, we give the definition of viscosity solution:

**Definition 11.1.** Let  $\Omega$  be a bounded domain of  $\mathbb{R}^n$  and let  $u$  be a continuous function in  $\Omega$ . We say that  $u$  is a viscosity solution in  $\Omega$  if

- i)  $\Delta u = 0$  in  $\Omega^+(u)$  and  $\Omega^-(u)$ ,
- ii) along the free boundary  $\Gamma$ ,  $u$  satisfies the free boundary condition, in the sense that:
  - a) if at  $x_0 \in \Gamma$  there exists a ball  $B \subset \Omega^+(u)$  such that  $x_0 \in \partial B$  and

$$u^+(x) \geq \alpha \langle x - x_0, \nu \rangle^+ + o(|x - x_0|), \quad \text{for } x \in B,$$

$$u^-(x) \leq \beta \langle x - x_0, \nu \rangle^- + o(|x - x_0|), \quad \text{for } x \in B^c,$$

for some  $\alpha > 0$  and  $\beta \geq 0$ , with equality along every non-tangential domain, then the free boundary condition is satisfied

$$G(\alpha, \beta) = 0,$$

- b) if at  $x_0 \in \Gamma$  there exists a ball  $B \subset \Omega^-(u)$  such that  $x_0 \in \partial B$  and

$$u^-(x) \geq \beta \langle x - x_0, \nu \rangle^- + o(|x - x_0|), \quad \text{for } x \in B,$$

$$u^+(x) \leq \alpha \langle x - x_0, \nu \rangle^+ + o(|x - x_0|), \quad \text{for } x \in \partial B,$$

for some  $\alpha \geq 0$  and  $\beta > 0$ , with equality along every non-tangential domain, then

$$G(\alpha, \beta) = 0.$$

**Lemma 11.2.** Let  $u$  be a minimizer in  $\Omega$  for the functional  $J$  in (1.8). Then,  $u$  is also a viscosity solution in the sense of Definition 11.1.

*Proof.* See Lemma 11.17 in [8] or Theorem 4.2 in [11] for the proof.  $\square$

Notice that, if  $x_0 \in \partial_{\text{red}}\{u > 0\}$ , then  $\Gamma$  is flat near  $x_0$ . Therefore, since a minimizer  $u$  is also a viscosity solution, according to Lemma 11.2, we can use the Harnack inequality approach to  $u$ , and obtain that  $\Gamma$  is  $C^{1,\alpha}$  in some neighborhood of  $x_0$ .

We next show that in two dimensions the free boundary is a continuously differentiable curve.

**11.2. The case in which  $u^-$  is degenerate.** In this section we show that near the points  $x_0 \in \Gamma$  where  $u^-$  is degenerate, the minimizer  $u$  behaves essentially as a solution to the one-phase problem. Recall that we say that  $u^-$  is degenerate at  $x_0$  if

$$(11.1) \quad \liminf_{r \rightarrow 0} \frac{1}{r} \int_{B_r(x_0)} u^- = 0.$$

We stress that  $u^+$  (in contrast to  $u^-$ ) is always nondegenerate, according to the following observation:

**Lemma 11.3.** Let  $u$  be a minimizer in  $\Omega$  for the functional  $J$  in (1.8), with  $0 \in \partial\{u > 0\}$ . Assume that (5.15) and (5.16) are satisfied. Then

$$\liminf_{r \rightarrow 0} \frac{1}{r} \int_{B_r} u^+ > 0.$$

*Proof.* By the clean ball condition in Theorem 8.1, we have that  $B_{c_1 r}(y_0) \subseteq B_r \cap \Omega^+(u)$ , for a suitable point  $y_0 \in B_r$  and a constant  $c_1 > 0$ .

Thus, from Lemma 5.4, we have that  $u(y) \geq c_2 r$  for any  $y \in B_{c_1 r/2}(y_0)$ , for some  $c_2 > 0$ . As a consequence,

$$\int_{B_r} u^+ \geq \int_{B_{c_1 r/2}(y_0)} u^+ \geq c_2 r \mathcal{L}^n(B_{c_1 r/2}(y_0)),$$

which gives the desired result.  $\square$

Next, we show that if  $u^-$  is degenerate then so is the gradient of  $u^-$  in the following sense:

**Lemma 11.4.** *Let  $u$  be a minimizer in  $\Omega$  for the functional  $J$  in (1.8). Let  $x_0 \in \partial\{u > 0\}$ , and suppose that  $u^-$  is degenerate at  $x_0$ . Then*

$$\lim_{r \rightarrow 0} \sup_{x \in B_r(x_0)} |\nabla u^-(x)| = 0.$$

*Proof.* We argue by contradiction and we suppose that the conclusion of the lemma fails. Then, there exists a sequence  $r_j \rightarrow 0$ , as  $j \rightarrow +\infty$ , such that

$$(11.2) \quad \lim_{j \rightarrow +\infty} \frac{1}{r_j} \int_{B_{r_j}(x_0)} u^- = 0$$

$$(11.3) \quad \text{and} \quad \lim_{j \rightarrow +\infty} \sup_{B_{r_j}(x_0)} |\nabla u^-| > 0.$$

Consider the scaled functions  $u_j(x) := \frac{u(x_0 + r_j x)}{r_j}$ . From Theorem 1.3 we obtain that  $|\nabla u_j|$  is bounded uniformly in  $j$  in  $B_1$ . Then, up to a subsequence, we have that  $u_j \rightarrow u_0$  as  $j \rightarrow +\infty$  uniformly in  $B_1$ , for some function  $u_0$ . Moreover, by (11.2),

$$0 = \lim_{j \rightarrow +\infty} \frac{1}{r_j} \int_{B_{r_j}(x_0)} u^- = \lim_{j \rightarrow +\infty} \int_{B_1} u_j^- = \int_{B_1} u_0^-.$$

This implies that

$$(11.4) \quad u_0^- = 0 \text{ in } B_1.$$

In addition, by (11.3) and Lemma 9.2,

$$0 < \lim_{j \rightarrow +\infty} \sup_{B_{r_j}(x_0)} |\nabla u^-| = \lim_{j \rightarrow +\infty} \sup_{B_1} |\nabla u_j^-| = \sup_{B_1} |\nabla u_0^-|,$$

which is in contradiction with (11.4).  $\square$

From Lemmata 11.3 and 11.4 it follows that, near a degenerate point  $x_0 \in \Gamma$ ,  $u$  behaves almost like a minimizer of a one-phase functional. It is well-known that for the one-phase the gradient  $|\nabla u|$  is upper semicontinuous. The aim of the next two lemmata is to establish this property of the gradient near degenerate points.

First we recall the Bernoulli constant

$$(11.5) \quad \Lambda(x_0) := \left[ \lambda_2 \partial_{r_2} \Phi \left( \lambda_1 \int_{\Omega} Q \chi_{\{u < 0\}}, \lambda_2 \int_{\Omega} Q \chi_{\{u > 0\}} \right) - \lambda_1 \partial_{r_1} \Phi \left( \lambda_1 \int_{\Omega} Q \chi_{\{u < 0\}}, \lambda_2 \int_{\Omega} Q \chi_{\{u > 0\}} \right) \right] Q(x_0)$$

measuring the gradient jump across the free boundary. We observe that, in view of Lemma 6.1 and using the notation in (6.1), if  $x_0$  is a smooth point of  $\partial\{u > 0\}$ , we know that

$$(11.6) \quad (\partial_\nu^+ u(x_0))^2 - (\partial_\nu^- u(x_0))^2 = \Lambda(x_0).$$

Furthermore, from Lemma 11.4 we should get that  $(\partial_\nu u^+(x))^2 = \Lambda(x_0) + o(1)$  as  $x \rightarrow x_0$ . The next lemma<sup>3</sup> makes this statement precise.

**Lemma 11.5.** *Let  $u$  be a minimizer in  $\Omega$  for the functional  $J$  in (1.8). Let  $x_0 \in \partial\{u > 0\}$ , and suppose that  $u^-$  is degenerate at  $x_0$ . Then,*

$$(11.7) \quad \limsup_{x \rightarrow x_0} |\nabla u(x)|^2 = \Lambda(x_0).$$

---

<sup>3</sup>We point out that Lemma 11.5 is not explicitly used anywhere in this paper, but we included it for completeness, since it clarifies the picture of the case in which  $u^-$  is degenerate.

*Proof.* Let us denote by

$$(11.8) \quad \gamma := \limsup_{x \rightarrow x_0} |\nabla u(x)|.$$

By Lemma 11.4,

$$\gamma = \limsup_{x \rightarrow x_0} |\nabla u^+(x)|,$$

hence, in order to prove (11.7), one has to show that

$$(11.9) \quad \gamma^2 = \Lambda(x_0).$$

For this, we take a sequence  $x_k \rightarrow x_0$  such that  $x_k \in \{u > 0\}$  and  $|\nabla u^+(x_k)| \rightarrow \gamma$  as  $k \rightarrow +\infty$ .

Let  $\rho_k := \text{dist}(x_k, \partial\{u > 0\})$  and let  $y_k \in \partial\{u > 0\}$  such that  $\rho_k = |x_k - y_k|$ . Consider the blow-up sequence  $u_k(x) := \frac{u(y_k + \rho_k x)}{\rho_k}$ . From Proposition 9.1, up to a subsequence, we may assume that  $u_k \rightarrow u_0$  as  $k \rightarrow +\infty$  locally uniformly.

Without loss of generality we can also assume that

$$\frac{x_k - y_k}{\rho_k} \rightarrow -e_n, \quad \text{as } k \rightarrow +\infty,$$

where  $e_n$  is the unit direction of the  $x_n$  axis. Thus we have that

$$(11.10) \quad B_1(-e_n) \subseteq \{u_0 > 0\}.$$

This and Lemma 9.2 give that

$$(11.11) \quad \gamma = \lim_{k \rightarrow +\infty} |\nabla u^+(x_k)| = \lim_{k \rightarrow +\infty} \left| \nabla u_k^+ \left( \frac{x_k - y_k}{\rho_k} \right) \right| = |\nabla u_0(-e_n)|.$$

From (11.10), we also obtain that

$$(11.12) \quad u_0 \text{ is harmonic in } B_1(-e_n),$$

thanks to Lemma 2.2 and Proposition 9.1.

We also observe that

$$(11.13) \quad |\nabla u_0| \leq \gamma \text{ in } B_1(-e_n).$$

Indeed, if  $\bar{x} \in B_1(-e_n)$ , we write  $\bar{x} = -e_n + z$ , with  $|z| < 1$  and we set

$$z_k := y_k + \rho_k \bar{x} = y_k + \rho_k z - \rho_k e_n = \rho_k \left( \frac{y_k - x_k}{\rho_k} - e_n \right) + \rho_k z + x_k \rightarrow x_0,$$

as  $k \rightarrow +\infty$ . Thus, by (11.8),

$$(11.14) \quad \gamma = \limsup_{x \rightarrow x_0} |\nabla u(x)| \geq \lim_{k \rightarrow +\infty} |\nabla u(z_k)|.$$

On the other hand,

$$\nabla u_k(\bar{x}) = \nabla u(y_k + \rho_k \bar{x}) = \nabla u(z_k).$$

Hence, taking the limit as  $k \rightarrow +\infty$  (and recalling (11.10) and Lemma 9.2), we see that

$$|\nabla u_0(\bar{x})| = \lim_{k \rightarrow +\infty} |\nabla u(z_k)|.$$

This and (11.14) imply (11.13), as desired.

We also remark that

$$(11.15) \quad \gamma > 0.$$

Indeed, if  $\gamma = 0$ , it follows from (11.13) that  $u_0$  is constant in  $B_1(-e_n)$ . Thus, since

$$(11.16) \quad u_0(0) = \lim_{k \rightarrow +\infty} u_k(0) = \lim_{k \rightarrow +\infty} \frac{u(y_k)}{\rho_k} = 0,$$

we obtain that  $u_0$  vanishes identically in  $B_1(-e_n)$ , in contradiction with (11.10), thus proving (11.15).

Now, we claim that

$$(11.17) \quad u_0(x) = -\nabla u_0(-e_n) \cdot x \text{ for any } x \in B_1(-e_n).$$

For this, we argue as follows: by (11.11) and (11.15), we can define

$$\ell := -\frac{\nabla u_0(-e_n)}{\gamma} = -\frac{\nabla u_0(-e_n)}{|\nabla u_0(-e_n)|}.$$

Then, by (11.11) and (11.13), we find that

$$(11.18) \quad \partial_\ell u_0(-e_n) = -\gamma \quad \text{and} \quad \partial_\ell u_0(x) \geq -\gamma \text{ in } B_1(-e_n).$$

Furthermore, in light of (11.12), we know that  $\partial_\ell u_0$  is harmonic in  $B_1(-e_n)$ . This, (11.18) and the strong maximum principle imply that  $\partial_\ell u_0 = -\gamma$  in  $B_1(-e_n)$ .

Then, we take a rotation  $\mathcal{R}$  such that  $e_1 = \mathcal{R}\ell$  and we define  $v_0(x) := u_0(\mathcal{R}x)$ . We have that

$$\partial_1 v_0(x) = (\mathcal{R}\nabla u_0(\mathcal{R}x)) \cdot e_1 = \nabla u_0(\mathcal{R}x) \cdot \ell = -\gamma$$

for any  $x$  such that  $\mathcal{R}x \in B_1(-e_n)$ .

Consequently, for any  $x$  such that  $\mathcal{R}x \in B_1(-e_n)$ , we have that

$$(11.19) \quad v_0(x) = -\gamma x_1 + \tilde{v}(x_2, \dots, x_n),$$

for some  $\tilde{v} : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ . In particular,

$$(11.20) \quad |\nabla v_0|^2 = |\gamma|^2 + \sum_{i=2}^n |\partial_i \tilde{v}|^2.$$

On the other hand, by (11.13), for any  $x$  such that  $\mathcal{R}x \in B_1(-e_n)$ ,

$$|\nabla v_0(x)|^2 = |\mathcal{R}\nabla u_0(\mathcal{R}x)|^2 \leq |\nabla u_0(\mathcal{R}x)|^2 \leq \gamma.$$

Then we insert this into (11.20) and we obtain that  $\partial_i \tilde{v}$  vanishes identically for any  $i \in \{2, \dots, n\}$ , hence  $\tilde{v}$  is constant, and (11.19) reduces to

$$v_0(x) = -\gamma x_1 + \tilde{c},$$

for some  $\tilde{c} \in \mathbb{R}$ .

Now, we recall (11.16) and we obtain that

$$0 = u_0(0) = v_0(0) = \tilde{c}.$$

As a consequence, we obtain that

$$v_0(x) = -\gamma x \cdot e_1 = -\gamma x \cdot \mathcal{R}^T \ell = -\gamma(\mathcal{R}x) \cdot \ell.$$

Hence

$$u_0(x) = v_0(\mathcal{R}^T x) = -\gamma x \cdot \ell,$$

thus completing the proof of (11.17).

Now, from (11.10) and (11.17), we deduce that  $\ell = e_n$ , and therefore

$$(11.21) \quad u_0(x) = -\gamma x_n \text{ in } B_1(-e_n).$$

We claim that, in fact,

$$(11.22) \quad u_0(x) = -\gamma x_n \text{ in } \{x_n < 0\}.$$

To check this, we recall (11.10) and we denote by  $\mathcal{C}$  the connected component of  $\{u_0 > 0\}$  that contains  $B_1(-e_n)$ . By Corollary 1.2, we know that  $u_0$  is harmonic in  $\mathcal{C}$ . Hence, by (11.21) and the unique continuation principle, we obtain that  $u_0(x) = -\gamma x_n$  in  $\mathcal{C}$ . As a consequence, since  $u_0$  vanishes along  $\partial\mathcal{C}$ , we have that

$$\partial\mathcal{C} \subseteq \{-\gamma x_n = 0\} = \{x_n = 0\},$$

thanks to (11.15), and this establishes (11.22).



It remains to show that  $\partial\{u_0 > 0\} = \{y_n = 0\}$ . To see this it is enough to show that there exists  $\delta > 0$  such that

$$(11.23) \quad u_0 = 0 \text{ in } \{x_n \in (0, \delta)\}.$$

Suppose that

$$s = \limsup_{\substack{y_n \searrow 0, \\ y' \in \mathbb{R}^{n-1}, \\ u_0(y', y_n) > 0}} \frac{\partial u_0(y', y_n)}{\partial y_n}.$$

Note that  $u_0$  is a minimizer of ACF functional, see Theorem 1.5. Taking a sequence  $\frac{\partial u_0(y'_k, h_k)}{\partial y_n} \rightarrow 0$  as  $h_k \rightarrow 0$  and using the same argument above it follows that the second blow of  $u_0$ , which we call  $u_{00}$ , with respect to the balls  $B_{h_k}(y'_k, 0)$  is of the form  $u_{00} = sy_n$ , with  $y_n > 0$ . This is a contradiction, since the zero set of the minimizers of ACF functional has nontrivial measure. Thus it follows that  $s = 0$  and consequently we have that  $u_0 = 0$  in some strip  $\{0 < y_n < \delta\}$ , for a suitable  $\delta > 0$ . This establishes (11.23).

Now, in light of (11.6), (11.22), and (11.23), we have that

$$\Lambda(x_0) = (\partial_\nu^+ u_0(0))^2 - (\partial_\nu^- u_0(0))^2 = \gamma^2 - 0,$$

which proves (11.9), as desired.  $\square$

Now we prove a flatness result in dimension 2 for the case in which  $u^-$  is degenerate.

**Proposition 11.6.** *Let  $u$  be as in Lemma 11.5. If  $n = 2$  then the free boundary of  $u$  is flat at every point.*

*Proof.* The proof is based on a compactness argument. Consider  $u_k(x) = \frac{u(x_0 + r_k x)}{r_k}$  for some positive sequence  $r_k \searrow 0$  with  $x_0 \in \partial\{u > 0\}$ .

Since, by Theorem 1.3,  $u$  is Lipschitz continuous, it follows that  $\{u_k\}$  is locally Lipschitz. So, in view of Proposition 9.1, we can employ a customary compactness argument to show that there exists a subsequence  $\{u_k\}$  converging to a limit  $u_0 \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^2)$  such that

$$(11.24) \quad u_k \rightarrow u_0 \quad \text{strongly in } W_{\text{loc}}^{1,p}(\mathbb{R}^2), \forall p > 1 \text{ and } C_{\text{loc}}^\alpha(\mathbb{R}^n), \forall \alpha \in (0, 1) \text{ as } k \rightarrow \infty,$$

$$(11.25) \quad \partial\{u_k > 0\} \rightarrow \partial\{u_0 > 0\} \quad \text{in Hausdorff distance } d_{\mathcal{H}} \text{ locally in } \mathbb{R}^2,$$

$$(11.26) \quad \chi_{\{u_k > 0\}} \rightarrow \chi_{\{u_0 > 0\}} \quad \text{in } L_{\text{loc}}^1(\mathbb{R}^2).$$

For the proofs of (11.24)-(11.26) we refer the reader for instance to [9] and [16].

Let  $u_0$  be a blow-up of  $u$  at  $x_0 \in \partial\{u > 0\}$ . Then, by Theorem 1.5 and the degeneracy of  $u^-$ , it holds that  $u_0 \geq 0$  is a minimizer of the Alt-Caffarelli functional in  $\mathbb{R}^2$  with the constant  $\lambda_0 = \lambda_0(u)$  given in (1.21). Hence, we now claim that

$$(11.27) \quad u_0 = \lambda_0 x_1^+$$

in an appropriate coordinate system (in which  $x_0$  is the origin). Indeed, let

$$W(x, r, u_0) = \frac{1}{r^2} \int_{B_r(x)} |\nabla u_0|^2 + \lambda_0^2 \chi_{\{u_0 > 0\}} - \frac{1}{r^4} \int_{\partial B_r(x)} u_0^2$$

be the Weiss energy. From the monotonicity of  $W$  (see [18]) it follows that if we first let  $r \rightarrow 0$ , and then  $r \rightarrow \infty$  for fixed  $s > 0$  then

$$W(0, s, u_{00}) \leq W(0, 0, u_0) \leq W(0, sr, u_0) \leq W(0, \infty, u_0) = W(0, s, u_{0\infty}),$$

where  $u_{00}$  is a blow-up of  $u_0$  at the origin and  $u_{0\infty}$  a blow-down of  $u_0$  at 0, namely

$$u_{0\infty} = \lim_{R_k \rightarrow \infty} \frac{u_0(R_k x)}{R_k}.$$

Both the blow-up and the blow-down are well-defined for  $u_0$  because it is a minimizer of the Alt-Caffarelli functional, see [18]. But  $W(0, s, u_{00}) = W(0, s, u_{0\infty}) = \text{const}$ , since in a suitable coordinate system both

functions  $u_{00}, u_{0\infty}$  will have the form  $\lambda_0 x_1^+$ . Hence  $u_0$  must be homogeneous and therefore linear, thus proving (11.27), which in turn gives the desired result.  $\square$

**11.3. The case in which  $u^-$  is nondegenerate.** If  $u^-$  is nondegenerate then the Alt-Caffarelli-Friedman functional

$$(11.28) \quad \phi(r, x_0, u) = \frac{1}{r^4} \int_{B_r(x_0)} \frac{|\nabla u^+|^2}{|x - x_0|^2} dx \int_{B_r(x_0)} \frac{|\nabla u^-|^2}{|x - x_0|^2} dx$$

has positive limit and therefore from Theorem 7.4 (i) in [3] when  $n = 2$  the blow-up  $u_0$  must be a two-plane solution.

**Lemma 11.7.** *Let  $u$  be a minimizer in  $\Omega$  for the functional  $J$  in (1.8), with  $0 \in \partial\{u > 0\}$ . Assume that (5.15) and (5.16) are satisfied.*

*Let  $\{r_j\}_{j \in \mathbb{N}}$  be a sequence of positive numbers such that  $r_j \searrow 0$  as  $j \rightarrow +\infty$ . Let  $x_0 \in \partial\{u > 0\}$  and set  $u_j(x) := \frac{u(x_0 + r_j x)}{r_j}$  such that  $u_j \rightarrow u_0$  as  $j \rightarrow +\infty$  for some subsequence, still denoted by  $\{r_j\}$ . If  $n = 2$  and  $u^-$  is nondegenerate at  $x_0$  then*

$$\lim_{r_j \rightarrow 0} \phi(r_j, x_0, u) = \gamma > 0$$

*and  $u_0$  is a two-plane solution, namely*

$$u_0(x) = \mu_1(x \cdot \ell)^+ - \mu_2(x \cdot \ell)^-$$

*for some unit direction  $\ell$  and positive constants  $\mu_1, \mu_2$ .*

*Proof.* From the scale invariance of the Alt-Caffarelli-Friedman functional we have that

$$\phi(r_j s, x_0, u) = \phi(s, 0, u_j).$$

Since, by Lemma 11.3, we have that  $u^+$  is nondegenerate, and by assumption so is  $u^-$  at  $x_0$ , then it follows that the limit

$$\lim_{r_j \rightarrow 0} \phi(sr_j, x_0, u)$$

exists and is independent of  $s > 0$ , because  $\phi$  is monotone and bounded thanks to Lipschitz continuity of  $u$ . Therefore we have that

$$\phi(s, 0, u_0) = \gamma > 0, \quad \forall s > 0,$$

which implies that  $u_0$  must be a homogeneous function of degree 1 (by the nondegeneracy of  $u_0^+$  and the Lipschitz regularity of  $u$ , recall Theorem 1.4 and Corollary 5.3). Applying Lemma 6.6 in [3], the desired result follows.  $\square$

Summarizing Proposition 11.6 and Lemma 11.7, we obtain the result in Theorem 1.7:

*Proof of Theorem 1.7.* By Lemma 11.2 we know that  $u$  is a viscosity solution. It follows from Proposition 11.6 and Lemma 11.7 that the free boundary  $\partial\{u > 0\}$  is flat at each point. Hence, the proof of the theorem follows from the regularity theory of Caffarelli developed for the viscosity solutions [6, 7]. See also Proposition 6.1 in [11].  $\square$

#### CONFLICT OF INTERESTS STATEMENT

The authors declare that they do not have any conflict of interests.

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